

A Floer homology for exact contact embeddings

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Abstract

In this paper we construct the Floer homology for an action functional which was introduced by Rabinowitz and prove a vanishing theorem. As an application, we show that there are no displaceable exact contact embeddings of the unit cotangent bundle of a sphere of dimension greater than three into a convex exact symplectic manifold with vanishing first Chern class. This generalizes Gromov's result that there are no exact Lagrangian embeddings of a sphere into \mathbb{C}^n .

Contents

1 Introduction

Exact convex symplectic manifolds and hypersurfaces. An *exact convex symplectic manifold* (V, λ) is a connected manifold V of dimension $2n$ without boundary with a one-form λ such that the following conditions are satisfied.

- (i) The two-form $\omega = d\lambda$ is symplectic.

- (ii) The symplectic manifold (V, ω) is convex at infinity, i.e. there exists an exhaustion $V = \cup_k V_k$ of V by compact sets $V_k \subset V_{k+1}$ with smooth boundary such that $\lambda|_{\partial V_k}$ is a contact form.

(cf. [13]). Define a vector field Y_λ on V by $i_{Y_\lambda} \omega = \lambda$. Then the last condition is equivalent to saying that Y_λ points out of V_k along ∂V_k .

We say that an exact convex symplectic manifold (V, λ) is *complete* if the vector field Y_λ is complete. We say that (V, λ) has *bounded topology* if $Y_\lambda \neq 0$ outside a compact set. Note that (V, λ) is complete and of bounded topology iff there exists an embedding $\phi : M \times \mathbb{R}_+ \rightarrow V$ such that $\phi^* \lambda = e^r \alpha_M$ with contact form $\alpha_M = \phi^* \lambda|_{M \times \{0\}}$, and such that $V \setminus \phi(M \times \mathbb{R}_+)$ is compact. (To see this, simply apply the flow of Y_λ to $M := \partial V_k$ for large k).

We say that a subset $A \subset V$ is *displaceable* if it can be displaced from itself via a Hamiltonian isotopy, i.e. there exists a smooth family of Hamiltonian functions $H = H(A) \in C^\infty([0, 1] \times V)$ with compact support such that the time one flow ϕ_H of the time dependent Hamiltonian vector field X_{H_t} defined by $dH_t = -\iota_{X_{H_t}} \omega$ for $H_t = H(t, \cdot) \in C^\infty(V)$ and $t \in [0, 1]$ satisfies $\phi_H(A) \cap A = \emptyset$.

The main examples of exact convex symplectic manifolds we have in mind are Stein manifolds. We briefly recall its definition. A *Stein manifold* is a triple (V, J, f) where V is a connected manifold, J is an integrable complex structure on V and $f \in C^\infty(V)$ is an exhausting plurisubharmonic function, i.e. f is proper and bounded from below, and the exact two form $\omega = -dd^c f$ is symplectic. Here the one form $\lambda = -d^c f$ is defined by the condition $d^c f(\xi) = df(J\xi)$ for every vector field ξ . We refer to [7] for a detailed treatment of Stein manifolds and Eliashberg's topological characterization of them. It is well known that if the plurisubharmonic function f is Morse, then all critical points of f have Morse index less than or equal than half the dimension of V , see for example [7]. The Stein manifold (V, J, f) is called *subcritical* if this inequality is strict. In a subcritical Stein manifold, every compact subset A is displaceable [3, Lemma 3.2].

Remark. Examples of exact convex symplectic manifolds which are not Stein can be obtained using the following construction. Let M be a $(2n-1)$ -dimensional closed manifold which admits a pair of contact forms (α_0, α_1) satisfying

$$\alpha_1 \wedge (d\alpha_1)^{n-1} = -\alpha_0 \wedge (d\alpha_0)^{n-1} > 0$$

and

$$\alpha_i \wedge (d\alpha_i)^k \wedge (d\alpha_j)^{n-k-1} = 0, \quad 0 \leq k \leq n-2$$

where (i, j) is a permutation of $(0, 1)$. Then a suitable interpolation of α_0 and α_1 endows the manifold $V = M \times [0, 1]$ with the structure of an exact convex symplectic manifold, where the restriction of the one-form to $M \times \{0\}$ is given by α_0 and the restriction to $M \times \{1\}$ is given by α_1 . Since $H_{2n-1}(V) = \mathbb{Z}$, the manifold V does not admit a Stein structure. Moreover, what makes these examples particularly interesting is the fact that they have two boundary components, whereas the boundary of a connected Stein manifold is always connected. The

first construction in dimension four of an exact convex symplectic manifold of the type above was carried out by D. McDuff in [30]. H. Geiges generalized her method in [21], where he also obtained higher dimensional examples.

If (V, λ) is an exact convex symplectic manifold then so is its *stabilization* $(V \times \mathbb{C}, \lambda \oplus \lambda_{\mathbb{C}})$ for the one form $\lambda_{\mathbb{C}} = \frac{1}{2}(x dy - y dx)$ on \mathbb{C} . Moreover, in $(V \times \mathbb{C}, \lambda \oplus \lambda_{\mathbb{C}})$ every compact subset A is displaceable. It is shown in [6] that each subcritical Stein manifold is Stein deformation equivalent to a split Stein manifold, i.e. a Stein manifold of the form $(V \times \mathbb{C}, J \times i, f + |z|^2)$ for a Stein manifold (V, J, f) .

Remark. If (V, λ) is an exact convex symplectic manifold, then so is $(V, \lambda + dh)$ for any smooth function $h : V \rightarrow \mathbb{R}$ with compact support. We call the 1-forms λ and $\lambda + dh$ *equivalent*. For all our considerations only the equivalence class of λ will be relevant.

An *exact convex hypersurface* in an exact convex symplectic manifold (V, λ) is a compact hypersurface (without boundary) $\Sigma \subset V$ such that

- (i) There exists a contact 1-form α on Σ such that $\alpha - \lambda|_{\Sigma}$ is exact.
- (ii) Σ is *bounding*, i.e. $V \setminus \Sigma$ consists of two connected components, one compact and one noncompact.

Remarks. (1) It follows that the volume form $\alpha \wedge (d\alpha)^{n-1}$ defines the orientation of Σ as boundary of the bounded component of $V \setminus \Sigma$.

(2) If Σ is an exact convex hypersurface in (V, λ) with contact form α , then there exists an equivalent 1-form $\mu = \lambda + dh$ on V such that $\alpha = \mu|_{\Sigma}$. To see this, extend α to a 1-form β on V . As $(\beta - \lambda)|_{\Sigma}$ is exact, there exists a function h on a neighbourhood U of Σ such that $\beta - \lambda = dh$ on U . Now simply extend h to a function with compact support on V and set $\mu := \lambda + dh$.

(3) If $H^1(\Sigma; \mathbb{R}) = 0$ condition (i) is equivalent to $d\alpha = \omega|_{\Sigma}$.

(4) Condition (ii) is automatically satisfied if $H_{2n-1}(V; \mathbb{Z}) = 0$, e.g. if V is a stabilization or a Stein manifold of dimension > 2 .

Floer homology. In the following we assume that (V, λ) is a complete exact convex symplectic manifold of bounded topology, and $\Sigma \subset V$ is an exact convex hypersurface with contact form α . We will define an invariant $HF(\Sigma, V)$ as the Floer homology of an action functional which was studied previously by Rabinowitz [37].

A *defining Hamiltonian for Σ* is a function $H \in C^\infty(V)$ which is constant outside of a compact set of V , whose zero level set $H^{-1}(0)$ equals Σ , and whose Hamiltonian vector field X_H defined by $dH = -\iota_{X_H}\omega$ agrees with the Reeb vector field R of α on Σ . Defining Hamiltonians exist since Σ is bounding, and they form a convex space.

Fix a defining Hamiltonian H and denote by $\mathcal{L} = C^\infty(\mathbb{R}/\mathbb{Z}, V)$ the free loop space of V . Rabinowitz' action functional

$$\mathcal{A}^H : \mathcal{L} \times \mathbb{R} \rightarrow \mathbb{R}$$

is defined as

$$\mathcal{A}^H(v, \eta) := \int_0^1 v^* \lambda - \eta \int_0^1 H(v(t)) dt, \quad (v, \eta) \in \mathcal{L} \times \mathbb{R}.$$

One may think of \mathcal{A}^H as the Lagrange multiplier functional of the unperturbed action functional of classical mechanics also studied in Floer theory to a mean value constraint of the loop. The critical points of \mathcal{A}^H satisfy

$$\left. \begin{aligned} \partial_t v(t) &= \eta X_H(v(t)), \quad t \in \mathbb{R}/\mathbb{Z}, \\ H(v(t)) &= 0. \end{aligned} \right\} \quad (1)$$

Here we used the fact that H is invariant under its Hamiltonian flow. Since the restriction of the Hamiltonian vector field X_H to Σ is the Reeb vector field, the equations (1) are equivalent to

$$\left. \begin{aligned} \partial_t v(t) &= \eta R(v(t)), \quad t \in \mathbb{R}/\mathbb{Z}, \\ v(t) &\in \Sigma, \quad t \in \mathbb{R}/\mathbb{Z}, \end{aligned} \right\} \quad (2)$$

i.e. v is a periodic orbit of the Reeb vector field on Σ with period η .¹

Theorem 1.1 *Under the above hypotheses, the Floer homology $HF(\mathcal{A}^H)$ is well-defined. Moreover, if H_s for $0 \leq s \leq 1$ is a smooth family of defining functions for exact convex hypersurfaces Σ_s , then $HF(\mathcal{A}^{H_0})$ and $HF(\mathcal{A}^{H_1})$ are canonically isomorphic.*

Hence the Floer homology $HF(\mathcal{A}^H)$ is independent of the choice of the defining function H for an exact convex hypersurface Σ , and the resulting invariant

$$HF(\Sigma, V) := HF(\mathcal{A}^H)$$

does not change under homotopies of exact convex hypersurfaces.

The next result is a vanishing theorem for the Floer homology $HF(\Sigma, V)$.

Theorem 1.2 *If Σ is displaceable, then $HF(\Sigma, V) = 0$.*

Remark. The action functional \mathcal{A}^H is also defined if $H^{-1}(0)$ is not exact convex. However, in this case the Floer homology $HF(\mathcal{A}^H)$ cannot in general be defined because the moduli spaces of flow lines will in general not be compact up to breaking anymore. The problem is that the Lagrange multiplier η may go to infinity. This phenomenon actually does happen as the counterexamples to the Hamiltonian Seifert conjecture show, see [23] and the literature cited therein.

Denote by c_1 the first Chern class of the tangent bundle of V (with respect to an ω -compatible almost complex structure and independent of this choice, see [32]).

¹The period η may be negative or zero. We refer in this paper to Reeb orbits moved backwards as Reeb orbits with negative period and to constant orbits as Reeb orbits of period zero.

Evaluation of c_1 on spheres gives rise to a homomorphism $I_{c_1} : \pi_2(V) \rightarrow \mathbb{Z}$. If I_{c_1} vanishes then the Floer homology $HF_*(\Sigma, V)$ can be \mathbb{Z} -graded with half integer degrees, i.e. $* \in 1/2 + \mathbb{Z}$.

The third result is a computation of the Floer homology for the unit cotangent bundle of a sphere.

Theorem 1.3 *Let (V, λ) be a complete exact convex symplectic manifold of bounded topology satisfying $I_{c_1} = 0$. Suppose that $\Sigma \subset V$ is an exact convex hypersurface with contact form α such that $(\Sigma, \ker \alpha)$ is contactomorphic to the unit cotangent bundle S^*S^n of the sphere of dimension $n \geq 4$ with its standard contact structure. Then*

$$HF_k(\Sigma, V) = \begin{cases} \mathbb{Z}_2, & k \in \{-n + \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, n - \frac{1}{2}\} + \mathbb{Z} \cdot (2n - 2), \\ 0, & \text{else.} \end{cases}$$

Applications and discussion. The following well-known technical lemma will allow us to remove completeness and bounded topology from the hypotheses of our corollaries.

Lemma 1.4 *Assume that Σ is an exact convex hypersurface in the exact convex symplectic manifold (V, λ) . Then V can be modified outside of Σ to an exact convex symplectic manifold $(\hat{V}, \hat{\lambda})$ which is complete and of bounded topology. If $I_{c_1} = 0$ for V the same holds for \hat{V} . If Σ is displaceable in V , then we can arrange that it is displaceable in \hat{V} as well.*

Proof: Let $V_1 \subset V_2 \dots$ be the compact exhaustion in the definition of an exact convex symplectic manifold. Since Σ is compact, it is contained in V_k for some k . The flow of Y_λ for times $r \in (-1, 0]$ defines an embedding $\phi : \partial V_k \times (-1, 0] \rightarrow V_k$ such that $\phi^* \lambda = e^r \lambda_0$, where $\lambda_0 = \lambda|_{\partial V_k}$. Now define

$$(\hat{V}, \hat{\lambda}) := (V_k, \lambda) \cup_\phi (\partial V_k \times (-1, \infty), e^r \lambda_0).$$

This is clearly complete and of bounded topology. The statement about I_{c_1} is obvious. If Σ is displaceable by a Hamiltonian isotopy generated by a compactly supported Hamiltonian $H : [0, 1] \times V \rightarrow \mathbb{R}$, we choose k so large that $\text{supp} H \subset [0, 1] \times V_k$ and apply the same construction. \square

As a first consequence of Theorem 1.2, we recover some known cases of the Weinstein conjecture, see [46], [20].

Corollary 1.5 *Every displaceable exact convex hypersurface Σ in an exact convex symplectic manifold (V, λ) carries a closed characteristic. In particular, this applies to all exact convex hypersurfaces in a subcritical Stein manifold, or more generally in a stabilization $V \times \mathbb{C}$.*

Proof: In view of Lemma 1.4, we may assume without loss of generality that (V, λ) is complete and of bounded topology. Then by Theorem 1.2 the Floer

homology $HF(\mathcal{A}^H)$ vanishes, where H is a defining function for Σ . On the other hand, the action functional \mathcal{A}^H always has critical points corresponding to the constant loops in Σ . So the vanishing of the Floer homology implies that there must also exist nontrivial solutions of (2), which are just closed characteristics, connected to constant loops by gradient flow lines of \mathcal{A}^H . \square

For further applications, the following notation will be convenient. An *exact contact embedding* of a closed contact manifold (Σ, ξ) into an exact convex symplectic manifold (V, λ) is an embedding $\iota: \Sigma \rightarrow V$ such that

- (i) There exists a 1-form α on Σ such that $\ker \alpha = \xi$ and $\alpha - \iota^* \lambda$ is exact.
- (ii) The image $\iota(\Sigma) \subset V$ is bounding.

In other words, $\iota(\Sigma) \subset V$ is an exact convex hypersurface with contact form $\iota_* \alpha$ which is contactomorphic (via ι^{-1}) to (Σ, ξ) .

Now Theorems 1.2 and 1.3 together with Lemma 1.4 immediately imply

Corollary 1.6 *Assume that $n \geq 4$ and there exists an exact contact embedding ι of S^*S^n into an exact convex symplectic manifold satisfying $I_{c_1} = 0$. Then $\iota(S^*S^n)$ is not displaceable.*

Since in a stabilization $V \times \mathbb{C}$ all compact subsets are displaceable, we obtain in particular

Corollary 1.7 *For $n \geq 4$ there does not exist an exact contact embedding of S^*S^n into a subcritical Stein manifold, or more generally, into the stabilization $(V \times \mathbb{C}, \lambda \oplus \lambda_{\mathbb{C}})$ of an exact convex symplectic manifold (V, λ) satisfying $I_{c_1} = 0$.*

Remark. If n is even then there are no smooth embeddings of S^*S^n into a subcritical Stein manifold by topological reasons, see Appendix C. However, at least for $n = 3$ and $n = 7$ there are no topological obstructions, see the discussion below.

If (V, J, f) is a Stein manifold with f a Morse function, P. Biran [2] defines the *critical coskeleton* as the union of the unstable manifolds (w.r. to ∇f) of the critical points of index $\dim V/2$. It is proved in [2] that every compact subset $A \subset V$ which does not intersect the critical coskeleton is displaceable. For example, in a cotangent bundle the critical coskeleton (after a small perturbation) is one given fibre. Thus Corollary 1.6 implies

Corollary 1.8 *Assume that $n \geq 4$ and there exists an exact contact embedding ι of S^*S^n into a Stein manifold (V, J, f) satisfying $I_{c_1} = 0$. Then $\iota(\Sigma)$ must intersect the critical coskeleton. In particular, the image of an exact contact embedding of S^*S^n into a cotangent bundle T^*Q must intersect every fibre.*

Remark. Let $\iota: L \rightarrow V$ be an *exact Lagrangian embedding* of L into V , i.e. such that $\iota^*\lambda$ is exact. Since by Weinstein's Lagrangian neighbourhood theorem [31, Theorem 3.33] a tubular neighbourhood of $\iota(L)$ can be symplectically identified with a tubular neighbourhood of the zero section of the cotangent bundle of L , we obtain an exact contact embedding of S^*L into V . Thus the last 3 corollaries generalize corresponding results about exact Lagrangian embeddings. For example, Corollary 1.7 generalizes (for spheres) the well-known result [24, 1, 3] that there exist no exact Lagrangian embeddings into subcritical Stein manifolds. Corollary 1.8 implies (cf. [2]) that an embedded Lagrangian sphere of dimension ≥ 4 in a cotangent bundle T^*Q must intersect every fibre.

Remark. Let us discuss Corollary 1.7 in the cases $n \leq 3$ that are not accessible by our method of proof. We always equip \mathbb{C}^n with the canonical 1-form $\lambda = \frac{1}{2} \sum_{i=1}^n (x_i dy_i - y_i dx_i)$.

$n = 1$: Any embedding of two disjoint circles into \mathbb{C} is an exact contact embedding of S^*S^1 , so Corollary 1.7 fails in this case.

$n = 2$: In this case Corollary 1.7 is true for purely topological reasons; we present various proofs in Appendix C.

$n = 3$: In this case Corollary 1.7 is true for subcritical Stein manifolds and can be proved using symplectic homology, see the last remark in this section.

Example. In this example we illustrate that the preceding results about exact contact embeddings are sensitive to the contact structure. Let $n = 3$ or $n = 7$. Then $S^*S^n \cong S^n \times S^{n-1}$ embeds into $\mathbb{R}^{n+1} \times S^{n-1}$. On the other hand $\mathbb{R}^{n+1} \times S^{n-1}$ is diffeomorphic to the subcritical Stein manifold $T^*S^{n-1} \times \mathbb{C}$, and identifying S^*S^n with a level set in $T^*S^{n-1} \times \mathbb{C}$ defines a contact structure ξ on S^*S^n . Thus (S^*S^n, ξ) has an exact contact embedding into a subcritical Stein manifold (in fact into \mathbb{C}^n) for $n = 3, 7$, whereas $(S^*S^7, \xi_{\text{st}})$ admits no such embedding by Corollary 1.7. In particular, we conclude

Corollary 1.9 *The two contact structures ξ and ξ_{st} on $S^*S^7 \cong S^7 \times S^6$ described above are not diffeomorphic.*

Remarks. (1) Corollary 1.9 also holds in the case $n = 3$, although our method does not apply there. Indeed, the contact structures ξ and ξ_{st} on S^*S^n for $n = 3, 7$ are distinguished by their cylindrical contact homology (see [45], [48]).

(2) The contact structures ξ and ξ_{st} on $S^3 \times S^2$ are homotopic as almost contact structures, i.e. as symplectic hyperplane distributions. This follows simply from the fact (see e.g. [22]) that on 5-manifolds almost contact structures are classified up to homotopy by their first Chern classes and $c_1(\xi) = c_1(\xi_{\text{st}}) = 0$. It would be interesting to know whether ξ and ξ_{st} on $S^7 \times S^6$ are also homotopic as almost contact structures. Here the first obstruction to such a homotopy vanishes because $c_3(\xi) = c_3(\xi_{\text{st}}) = 0$, but there are further obstructions in dimensions 7 and 13 which remain to be analysed along the lines of [34].

Remark (obstructions from symplectic field theory). Symplectic field theory [12] also yields obstructions to exact contact embeddings. For example, by neck stretching along the image of an exact contact embedding, the following result is proved in [8]: *Let (Σ^{2n-1}, ξ) be a closed contact manifold with $H_1(\Sigma; \mathbb{Z}) = 0$ which admits an exact contact embedding into \mathbb{C}^n . Then for every nondegenerate contact form defining ξ there exist closed Reeb orbits of Conley-Zehnder indices $n + 1 + 2k$ for all integers $k \geq 0$.*

Here Conley-Zehnder indices are defined with respect to trivializations extending over spanning surfaces. This result applies in particular to the unit cotangent bundle $\Sigma = S^*Q$ of a closed Riemannian manifold Q with $H_1(Q; \mathbb{Z}) = 0$. For example, if Q carries a metric of nonpositive curvature then all indices are $\leq n - 1$ and hence S^*Q admits no exact contact embedding into \mathbb{C}^n . On the other hand, any nondegenerate metric on the sphere S^n has closed geodesics of all indices $n + 1 + 2k$, $k \geq 0$, so this result does *not* exclude exact contact embeddings $S^*S^n \hookrightarrow \mathbb{C}^n$.

Remark (obstructions from symplectic homology). Corollary 1.7 for subcritical Stein manifolds can be proved for all $n \geq 3$ by combining the following five results on symplectic homology. See [9] for details.

- (1) The symplectic homology $SH(V)$ of a subcritical Stein manifold V vanishes [5].
- (2) If $\Sigma \subset V$ is an exact convex hypersurface in an exact convex symplectic manifold bounding the compact domain $W \subset V$, then $SH(V) = 0$ implies $SH(W) = 0$. This follows from an argument by M. McLean [33], based on Viterbo's transfer map [46] and the ring structure on symplectic homology.
- (3) If $SH(W) = 0$, then the positive action part $SH^+(W)$ of symplectic homology is only nonzero in finitely many degrees. This follows from the long exact sequence induced by the action filtration.
- (4) $SH^+(W)$ equals the non-equivariant linearized contact homology $NCH(W)$. This is implicit in [4], see also [9].
- (5) If $\partial W = S^*S^n$ and $n \geq 3$, then $NCH(W)$ is independent of the exact filling W and equals the homology of the free loop space of S^n (modulo the constant loops), which is nonzero in infinitely many degrees.

2 Exact contact embeddings

Let Σ be a connected closed $2n - 1$ dimensional manifold. A *contact structure* ξ is a field of hyperplanes $\xi \subset T\Sigma$ such that there exists a one-form α satisfying

$$\xi = \ker \alpha, \quad \alpha \wedge d\alpha^{n-1} > 0.$$

The one form α is called a contact form. It is determined by ξ up to multiplication with a function $f > 0$. Given a contact form α the *Reeb vector field* R

on Σ is defined by the conditions

$$\iota_R d\alpha = 0, \quad \alpha(R) = 1.$$

Unit cotangent bundles have a natural contact structure as the following example shows.

Example. For a manifold N we denote by S^*N the oriented projectivization of its cotangent bundle T^*N , i.e. elements of S^*N are equivalence classes $[v^*]$ of cotangent vectors $v^* \in T^*N$ under the equivalence relation $v^* \cong w^*$ if there exists $r > 0$ such that $v^* = rw^*$. Denote by $\pi: S^*N \rightarrow N$ the canonical projection. A contact structure ξ on S^*N is given by

$$\xi_{[v^*]} = \ker v^* \circ d\pi([v^*]).$$

If g is a Riemannian metric on N then S^*N can be identified with the space of tangent vectors of N of length one and the restriction of the Liouville one form defines a contact form. Observe that the Reeb vector field generates the geodesic flow on N .

If $\iota: \Sigma \rightarrow V$ is a exact contact embedding, then $\alpha = \iota^*\lambda$ defines a contact form for the contact structure ξ . One might ask which contact forms α can arise in this way. The following proposition shows that if one contact form defining the contact structure ξ arises from an exact contact embedding, then every other contact form defining ξ arises as well.

Proposition 2.1 *Assume that $\iota: (\Sigma, \xi) \rightarrow (V, \lambda)$ is an exact contact embedding with $\xi = \ker \iota^*\lambda$. Then for every contact form α defining the contact structure ξ on Σ there exists a constant $c > 0$ and a bounding embedding $\iota_\alpha: \Sigma \rightarrow V$ such that $\iota_\alpha^*\lambda = c\alpha$.*

Proof of Proposition 2.1: The proof uses the fact that if there exists an exact contact embedding of a contact manifold into an exact convex symplectic manifold (V, λ) then the negative symplectization can be embedded. To see that we need two facts. Recall that the vector field Y_λ is defined by the condition $\lambda = \iota_{Y_\lambda} d\lambda$.

Fact 1: The flow ϕ_λ^t of Y_λ exists for all negative times t .

Indeed, let $x \in V$. Then $x \in V_k$ for some k . As Y_λ points out of V_k along ∂V_k , $\phi_\lambda^t(x) \in V_k$ for all $t \leq 0$ and compactness of V_k yields completeness for $t \leq 0$.

Fact 2: The vector field Y_λ satisfies

$$\iota_{Y_\lambda} \lambda = 0, \quad L_{Y_\lambda} \lambda = \lambda, \tag{3}$$

where L_{Y_λ} is the Lie derivative along the vector field Y_λ . In particular, the flow ϕ_λ^r of Y_λ satisfies $(\phi_\lambda^r)^*\lambda = e^r \lambda$.

Indeed, the first equation in (3) follows directly from the definition of Y_λ . To prove the second one we compute using Cartan's formula

$$L_{Y_\lambda} \lambda = d\iota_{Y_\lambda} \lambda + \iota_{Y_\lambda} d\lambda = \lambda.$$

Now set $\alpha_0 = \iota^* \lambda$ and consider the symplectic manifold $(\Sigma \times \mathbb{R}_-, d(e^r \alpha_0))$ where r denotes the coordinate on the \mathbb{R} -factor. By Fact 1, the flow ϕ_λ^r exists for all $r \leq 0$. By Fact 2, the embedding

$$\hat{\iota}: \Sigma \times \mathbb{R}_- \rightarrow V, \quad (x, r) \mapsto \phi_\lambda^r(\iota(x))$$

satisfies

$$(\hat{\iota})^* \lambda = e^r \alpha_0.$$

If α is another contact form on Σ which defines the contact structure ξ then there exists a smooth function $\rho_\alpha \in C^\infty(\Sigma)$ such that

$$\alpha = e^{\rho_\alpha} \alpha_0.$$

Set $m := \max_\Sigma \rho_\alpha$ and $c := e^{-m}$. Then

$$\iota_\alpha: \Sigma \rightarrow V, \quad x \mapsto \hat{\iota}(x, \rho_\alpha(x) - m)$$

gives the required contact embedding for α . This proves the proposition. \square

Remark. If the vector field Y_λ is complete, then the preceding proof yields a symplectic embedding of the whole symplectization $(\Sigma \times \mathbb{R}, d(e^r \alpha_0))$ into (V, ω) .

3 Floer homology for Rabinowitz's action functional

In this section we construct the Floer homology for Rabinowitz's action functional defined in the introduction and prove Theorem 1.1 and Theorem 1.2. We assume that the reader is familiar with the constructions in Floer theory which can be found in Floer's original papers [14, 15, 16, 17, 18] or in Salamon's lectures [42]. The finite dimensional case of Morse theory is treated in the book of Schwarz [44].

Throughout this section we maintain the following setup:

- (V, λ) is a complete exact convex symplectic manifold of bounded topology.
- $\Sigma \subset V$ is an exact convex hypersurface with contact form α and defining Hamiltonian H .

Our sign conventions for Floer homology are as follows:

- The *Hamiltonian vector field* X_H is defined by $dH = -i_{X_H} \omega$, where $\omega = d\lambda$.
- An almost complex structure J on V is ω -compatible if $\omega(\cdot, J\cdot)$ defines a Riemannian metric. Thus the gradient with respect to this metric is related to the symplectic vector field by $X_H = J\nabla H$.

- Floer homology is defined using the *positive* gradient flow of the action functional \mathcal{A}^H .

The action functional \mathcal{A}^H is invariant under the S^1 -action on $\mathcal{L} \times \mathbb{R}$ given by $t_*(v(\cdot), \eta) \mapsto (v(t + \cdot), \eta)$. In particular, the action functional \mathcal{A}^H will never be Morse. However, generically it is *Morse-Bott*, i.e. its critical set is a manifold whose tangent space is the kernel of the Hessian of the action functional. We make the following nondegeneracy assumption on the Reeb flow ϕ_t of the contact form α on Σ .

- (A) The closed Reeb orbits of (Σ, α) are of *Morse-Bott type*, i.e. for each $T \in \mathbb{R}$ the set $\mathcal{N}_T \subset \Sigma$ formed by the T -periodic Reeb orbits is a closed submanifold, the rank of $d\alpha|_{\mathcal{N}_T}$ is locally constant, and $T_p\mathcal{N}_T = \ker(T_p\phi_T - \text{id})$ for all $p \in \mathcal{N}_T$.

If the assumption (A) does not hold we consider a hypersurface close by. Note that the contact condition is an open condition and the assumption (A) is generically satisfied. Since we prove that our homology is invariant under homotopies we can assume without loss of generality that (A) holds. If (A) is satisfied, then the action functional \mathcal{A}^H is Morse-Bott.

Remark. Generically, we can even achieve that all T -periodic Reeb orbits γ with $T \neq 0$ are *nondegenerate*, i.e. the linearization $T_p\phi_T : \xi_p \rightarrow \xi_p$ at $p = \gamma(0)$ does not have 1 in its spectrum. In this case the critical manifold of \mathcal{A}^H consists of a union of circles for each nonconstant Reeb orbit and a copy of the hypersurface Σ for the constant solutions, i.e. critical points with $\eta = 0$. Moreover, observe that a nonconstant Reeb orbit gives rise to infinitely many of them because it can be repeatedly passed and also be passed in the backward direction.

There are several ways to deal with Morse-Bott situations in Floer homology. One possibility is to choose an additional small perturbation to get a Morse situation. This was carried out by Pozniak [36], where it was also shown that the local Floer homology near each critical manifold coincides with the Morse homology of the critical manifold. Another possibility is to choose an additional Morse function on the critical manifold. The chain complex is then generated by the critical points of this Morse function while the boundary operator is defined by counting flow lines with cascades. This approach was carried out by the second named author in [19].

Cascades are finite energy gradient flow lines of the action functional \mathcal{A}^H . In the Morse-Bott case the finite energy assumption is equivalent to assume that the gradient flow line converges at both ends exponentially to a point on the critical manifold. In order to prove that the Floer homology is well defined one has to show that the moduli spaces of cascades are compact modulo breaking. There are three difficulties one has to solve.

- An L^∞ -bound on the loop $v \in \mathcal{L}$.
- An L^∞ -bound on the Lagrange multiplier $\eta \in \mathbb{R}$.

- An L^∞ -bound on the derivatives of the loop v .

Although the first and the third point are nontrivial they are standard problems in Floer theory one knows how to deal with. The L^∞ -bound for the loop follows from the convexity assumption on V and the derivatives can be controlled since our symplectic manifold is exact and hence there is no bubbling of pseudo-holomorphic spheres. The new feature is the bound on the Lagrange multiplier η . We will explain in detail how this can be achieved. It will be essential that our hypersurface is convex.

We first explain the metric on the space $\mathcal{L} \times \mathbb{R}$ and deduce from that the equation for the cascades. The metric on $\mathcal{L} \times \mathbb{R}$ is the product metric of the standard metric on \mathbb{R} and a metric on \mathcal{L} coming from a family of ω -compatible almost complex structures J_t on V . For such a family of ω -compatible almost complex structures J_t we define the metric g_J on $\mathcal{L} \times \mathbb{R}$ for $(v, \eta) \in \mathcal{L} \times \mathbb{R}$ and $(\hat{v}_1, \hat{\eta}_1), (\hat{v}_2, \hat{\eta}_2) \in T_{(v, \eta)}(\mathcal{L} \times \mathbb{R})$ by

$$g_J((\hat{v}_1, \hat{\eta}_1), (\hat{v}_2, \hat{\eta}_2)) = \int_0^1 \omega(\hat{v}_1, J_t(v)\hat{v}_2)dt + \hat{\eta}_1 \cdot \hat{\eta}_2.$$

The gradient of \mathcal{A}^H with respect to this metric is given by

$$\nabla \mathcal{A}^H = \nabla_{g_J} \mathcal{A}^H = \begin{pmatrix} -J_t(v)(\partial_t v - \eta X_H(v)) \\ -\int_0^1 H(v(\cdot, t))dt \end{pmatrix}.$$

Thus gradient flow lines of $\nabla \mathcal{A}^H$ are solutions $(v, \eta) \in C^\infty(\mathbb{R} \times S^1, V \times \mathbb{R})$ of the following problem

$$\left. \begin{aligned} \partial_s v + J_t(v)(\partial_t v - \eta X_H(v)) &= 0 \\ \partial_s \eta + \int_0^1 H(v(\cdot, t))dt &= 0. \end{aligned} \right\} \quad (4)$$

The following proposition is our main tool to bound the Lagrange multiplier η .

Proposition 3.1 *There exists $\epsilon > 0$ such that for every $M > 0$ there exists a constant $c_M < \infty$ such that*

$$\left\{ \begin{aligned} \|\nabla \mathcal{A}^H(v, \eta)\| &\leq \epsilon \\ |\mathcal{A}^H(v, \eta)| &\leq M \end{aligned} \right\} \implies |\eta| \leq c_M.$$

We first prove a lemma which says that the action value of a critical point of \mathcal{A}^H , i.e. a Reeb orbit, is given by the period.

Lemma 3.2 *Let $(v, \eta) \in \text{crit}(\mathcal{A}^H)$, then $\mathcal{A}^H(v, \eta) = \eta$.*

Proof: Inserting (2) into \mathcal{A}^H we compute

$$\mathcal{A}^H(v, \eta) = \eta \int_0^1 \lambda(v)R(v) = \eta \int_0^1 \alpha(v)R(v) = \eta.$$

This proves the lemma. \square

Proof of Proposition 3.1: We prove the proposition in three steps. The first step is an elaboration of the observation in Lemma 3.2.

Step 1: *There exists $\delta > 0$ and a constant $c_\delta < \infty$ with the following property. For every $(v, \eta) \in \mathcal{L} \times \mathbb{R}$ such that $v(t) \in U_\delta = H^{-1}((-\delta, \delta))$ for every $t \in \mathbb{R}/\mathbb{Z}$, the following estimate holds:*

$$|\eta| \leq 2|\mathcal{A}^H(v, \eta)| + c_\delta \|\nabla \mathcal{A}^H(v, \eta)\|.$$

Choose $\delta > 0$ so small such that

$$\lambda(x)X_H(x) \geq \frac{1}{2} + \delta, \quad x \in U_\delta.$$

Set

$$c_\delta = 2\|\lambda|_{U_\delta}\|_\infty.$$

We estimate

$$\begin{aligned} |\mathcal{A}^H(v, \eta)| &= \left| \int_0^1 \lambda(v) \partial_t v - \eta \int_0^1 H(v(t)) dt \right| \\ &= \left| \eta \int_0^1 \lambda(v) X_H(v) + \int_0^1 \lambda(v) (\partial_t v - \eta X_H(v)) - \eta \int_0^1 H(v(t)) dt \right| \\ &\geq \left| \eta \int_0^1 \lambda(v) X_H(v) \right| - \left| \int_0^1 \lambda(v) (\partial_t v - \eta X_H(v)) \right| \\ &\quad - \left| \eta \int_0^1 H(v(t)) dt \right| \\ &\geq |\eta| \left(\frac{1}{2} + \delta \right) - \frac{c_\delta}{2} \|\partial_t v - \eta X_H(v)\|_1 - |\eta| \delta \\ &\geq \frac{|\eta|}{2} - \frac{c_\delta}{2} \|\partial_t v - \eta X_H(v)\|_2 \\ &\geq \frac{|\eta|}{2} - \frac{c_\delta}{2} \|\nabla \mathcal{A}^H(v, \eta)\|. \end{aligned}$$

This proves Step 1.

Step 2: *For each $\delta > 0$ there exists $\epsilon = \epsilon(\delta) > 0$ such that if $\|\nabla \mathcal{A}^H(v, \eta)\| \leq \epsilon$, then $v(t) \in U_\delta$ for every $t \in [0, 1]$.*

Denote by Γ_δ the set of smooth paths $\gamma \in C^\infty([0, 1], U_\delta)$ such that $|H(\gamma(0))| = \delta$ and $|H(\gamma(1))| = \delta/2$. For each $x \in U_\delta$ there is a splitting $T_x M = T_x H^{-1}(H(x)) \oplus T_x^\perp H^{-1}(H(x))$. We denote by π_x the projection to the second factor. We introduce the number $\epsilon_0 = \epsilon_0(\delta)$ by

$$\epsilon_0 = \inf_{\gamma \in \Gamma_\delta} \left\{ \int_0^1 \|\pi_{\gamma(t)}(\dot{\gamma}(t))\| dt \right\} > 0.$$

Now assume that $v \in \mathcal{L}$ has the property that there exist $t_0, t_1 \in \mathbb{R}/\mathbb{Z}$ such that $|H(v(t_0))| \geq \delta$ and $|H(v(t_1))| \leq \delta/2$. We claim that

$$\|\nabla \mathcal{A}^H(v, \eta)\| \geq \epsilon_0 \quad (5)$$

for every $\eta \in \mathbb{R}$. To see that we estimate

$$\begin{aligned} \|\nabla \mathcal{A}^H(v, \eta)\| &\geq \sqrt{\int_0^1 \|\partial_t v - \eta X_H(v)\|^2 dt} \\ &\geq \sqrt{\int_0^1 \|\pi_v(\partial_t v - \eta X_H(v))\|^2 dt} \\ &= \sqrt{\int_0^1 \|\pi_v(\partial_t v)\|^2 dt} \\ &\geq \int_0^1 \|\pi_v(\partial_t v)\| dt \\ &\geq \epsilon_0. \end{aligned}$$

This proves (5).

Now assume that $v \in \mathcal{L}$ has the property that $v(t) \in M \setminus U_{\delta/2}$ for every $t \in [0, 1]$. In this case we estimate

$$\|\nabla \mathcal{A}^H(v, \eta)\| \geq \left| \int_0^1 H(v(t)) dt \right| \geq \frac{\delta}{2} \quad (6)$$

for every $\eta \in \mathbb{R}$. From (5) and (6) Step 2 follows with $\epsilon < \min\{\epsilon_0, \delta/2\}$.

Step 3: *We prove the proposition.*

Combining Step 1 and Step 2 the proposition follows with $c_M = 2M + \epsilon c_\delta$. \square

Proposition 3.1 allows us to control the size of the Lagrange multiplier η . Our first corollary considers the case of gradient flow lines.

Corollary 3.3 *Assume that $(v, \eta) \in C^\infty(\mathbb{R} \times S^1, V) \times C^\infty(\mathbb{R}, \mathbb{R})$ is a gradient flow line of $\nabla \mathcal{A}^H$ which satisfies $\lim_{s \rightarrow \pm\infty} (v, \eta)(s, \cdot) = (v^\pm, \eta^\pm)(\cdot) \in \text{crit}(\mathcal{A}^H)$, where the limit is uniform in the t -variable. Then the L^∞ -norm of η is bounded uniformly in terms of a constant $c < \infty$ which only depends on $\mathcal{A}^H(v^-, \eta^-)$ and $\mathcal{A}^H(v^+, \eta^+)$.*

To prove invariance of our Floer homology under homotopies we also have to consider the case of s -dependent action functionals. Let $H^-, H^+ \in C^\infty(V)$ be defining Hamiltonians for two exact convex hypersurfaces. Consider the smooth family of s -dependent Hamiltonians H_s defined as

$$H_s = \beta(s)H^+ + (1 - \beta(s))H^-$$

where $\beta \in C^\infty(\mathbb{R}, [0, 1])$ is a smooth monotone increasing cutoff function such that $\beta(s) = 1$ for $s \geq 1$ and $\beta(s) = 0$ for $s \leq 0$.

Corollary 3.4 *If $\max_{x \in V} |H^+(x) - H^-(x)|$ is small enough, then for each gradient flow line (v, η) of the s -dependent action functional \mathcal{A}^{H_s} which converges at both ends the Lagrange multiplier η is uniformly bounded in terms of the action values at the end points.*

Proof of Corollary 3.3: Let ϵ be as in Proposition 3.1. For $\sigma \in \mathbb{R}$ let $\tau(\sigma) \geq 0$ be defined by

$$\tau(\sigma) := \inf \{ \tau \geq 0 : \|\nabla \mathcal{A}^H((v, \eta)(\sigma + \tau(\sigma)))\| < \epsilon \}.$$

We abbreviate the energy of the flow line (v, η) by

$$E := \mathcal{A}^H(v^+, \eta^+) - \mathcal{A}^H(v^-, \eta^-).$$

We claim that

$$\tau(\sigma) \leq \frac{E}{\epsilon^2}. \quad (7)$$

To see this we estimate

$$\begin{aligned} E &= \mathcal{A}^H(v^+, \eta^+) - \mathcal{A}^H(v^-, \eta^-) \\ &= \int_{-\infty}^{\infty} \frac{d}{ds} \mathcal{A}^H(v, \eta) ds \\ &= \int_{-\infty}^{\infty} d\mathcal{A}^H(v, \eta) \partial_s(v, \eta) ds \\ &= \int_{-\infty}^{\infty} \langle \nabla \mathcal{A}^H(v, \eta), \partial_s(v, \eta) \rangle ds \\ &= \int_{-\infty}^{\infty} \|\nabla \mathcal{A}^H(v, \eta)\|^2 ds \\ &\geq \int_{\sigma}^{\sigma + \tau(\sigma)} \|\nabla \mathcal{A}^H(v, \eta)\|^2 ds \\ &\geq \tau(\sigma) \epsilon^2. \end{aligned}$$

This implies (7).

We set

$$M := \max\{|\mathcal{A}^H(v^+, \eta^+)|, |\mathcal{A}^H(v^-, \eta^-)|\}.$$

Note that since the action increases along a gradient flow line we have

$$|\mathcal{A}^H((v, \eta)(\sigma + \tau(\sigma)))| \leq M \text{ for all } \sigma \in \mathbb{R}.$$

We deduce from Proposition 3.1 and the definition of $\tau(\sigma)$ that

$$|\eta(\sigma + \tau(\sigma))| \leq c_M. \quad (8)$$

We set

$$c_H := \max_{x \in V} |H(x)|. \quad (9)$$

We estimate using (7), (8), and (9)

$$\begin{aligned}
|\eta(\sigma)| &\leq |\eta(\sigma + \tau(\sigma))| + \int_{\sigma}^{\sigma + \tau(\sigma)} |\partial_s \eta(s)| ds \\
&= |\eta(\sigma + \tau(\sigma))| + \int_{\sigma}^{\sigma + \tau(\sigma)} \left| \int_0^1 H(v(s, t)) dt \right| ds \\
&\leq c_M + c_H \tau(\sigma) \\
&\leq c_M + \frac{c_H E}{\epsilon^2}.
\end{aligned}$$

The right hand side is independent of σ and hence we get

$$\|\eta\|_{\infty} \leq c_M + \frac{c_H E}{\epsilon^2}. \quad (10)$$

This proves the corollary. \square

Proof of Corollary 3.4: In the s -dependent case we define the energy as

$$E := \mathcal{A}^{H^+}(v^+, \eta^+) - \mathcal{A}^{H^-}(v^-, \eta^-) - \int_{-\infty}^{\infty} (\partial_s \mathcal{A}^{H_s})(v, \eta) ds,$$

where

$$(\partial_s \mathcal{A}^{H_s})(v, \eta) = -\eta \int_0^1 \frac{\partial H_s}{\partial s}(v) dt = -\eta \int_0^1 \beta'(s)(H^+ - H^-)(v) dt.$$

If we set $c_H := \max\{c_{H^+}, c_{H^-}\}$ and $\epsilon := \min\{\epsilon(H^+), \epsilon(H^-)\}$ then (10) can be deduced as in the time-independent case. However, E is a priori not bounded anymore because of the term containing the s -derivatives of the action functional. We use the abbreviations

$$\Delta := \mathcal{A}^{H^+}(v^+, \eta^+) - \mathcal{A}^{H^-}(v^-, \eta^-)$$

and

$$\delta := \max_{x \in V} |H^+(x) - H^-(x)|$$

and estimate

$$\begin{aligned}
E &= \Delta - \int_{-\infty}^{\infty} (\partial_s \mathcal{A}^{H_s})(v, \eta) ds \\
&= \Delta + \int_{-\infty}^{\infty} \beta'(s) \eta(s) \left(\int_0^1 (H^+ - H^-)(v(s, t)) dt \right) ds \\
&\leq \Delta + \delta \|\eta\|_{\infty}.
\end{aligned}$$

If we set this estimate into (10) we obtain

$$\|\eta\|_{\infty} \leq c_M + \frac{c_H \Delta}{\epsilon^2} + \frac{c_H \delta}{\epsilon^2} \|\eta\|_{\infty}.$$

Now if

$$\delta < \frac{\epsilon^2}{c_H}$$

we obtain the following uniform L^∞ -bound for η

$$\|\eta\|_\infty \leq \frac{\epsilon^2 c_M + c_H \Delta}{\epsilon^2 - c_H \delta}.$$

This proves Corollary 3.4. \square

Proof of Theorem 1.1: As we pointed out at the beginning of this section, we may assume without loss of generality that \mathcal{A}^H is Morse-Bott. Choose an additional Morse function h on $\text{crit}(\mathcal{A}^H)$. The Floer chain complex is defined in the following way. $CF(\mathcal{A}^H, h)$ is the \mathbb{Z}_2 -vector space consisting of formal sums

$$\xi = \sum_{c \in \text{crit}(h)} \xi_c c$$

where the coefficients $\xi_c \in \mathbb{Z}_2$ satisfy the finiteness condition

$$\#\{c \in \text{crit}(h) : \xi_c \neq 0, \mathcal{A}^H(c) \leq \kappa\} < \infty \quad (11)$$

for every $\kappa \in \mathbb{R}$. To define the boundary operator, we require some compatibility condition of the family of ω -compatible almost complex structures J_t with the convex structure of V at infinity in order to make sure that our cascades remain in a compact subset of V . As we remarked in the introduction, completeness implies that there exists a contact manifold (M, α_M) ² such that a neighbourhood of infinity of the symplectic manifold (V, ω) can be symplectically identified with $(M \times \mathbb{R}_+, d(e^r \alpha_M))$, where r refers to the coordinate on $\mathbb{R}_+ = \{r \in \mathbb{R} : r \geq 0\}$. We may assume without loss of generality that H is constant on $M \times \mathbb{R}_+$. We require the following conditions on J_t for every $t \in [0, 1]$.

- For each $x \in M$ we have $J_t(x) \frac{\partial}{\partial r} = R_M$, where R_M is the Reeb vector field on (M, α_M) .
- J_t leaves the kernel of α_M invariant for every $x \in M$.
- J_t is invariant under the local half flow $(x, 0) \mapsto (x, r)$ for $(x, r) \in M \times \mathbb{R}_+$.

We choose further an additional Riemannian metric g_c on the critical manifold $\text{crit}(\mathcal{A}^H)$. For two critical points $c_-, c_+ \in \text{crit}(h)$ we consider the moduli space of *gradient flow lines with cascades* $\mathcal{M}_{c_-, c_+}(\mathcal{A}^H, h, J, g_c)$ as defined in Appendix A. For generic choice of J and g_c this moduli space of is a smooth manifold. We claim that its zero dimensional component $\mathcal{M}_{c_-, c_+}^0(\mathcal{A}^H, h, J, g_c)$ is actually compact and hence a finite set. To see that we have to prove that cascades are compact modulo breaking. Since the support of X_H lies outside of $M \times \mathbb{R}_+$, the

²Be careful! Do not confuse the contact manifolds (M, α_M) and (Σ, α) .

first component of a gradient flow line which enters $M \times \mathbb{R}_+$ will just satisfy the pseudo-holomorphic curve equation by (4). By our choice of the family of almost complex structures the convexity condition guarantees that it cannot touch any level set $M \times \{r\}$ from inside (see [30]), and since its asymptotics lie outside of $M \times \mathbb{R}_+$ it has to remain in the compact set $V \setminus M \times \mathbb{R}_+$ all the time. This gives us a uniform L^∞ -bound on the first component. Corollary 3.3 implies that the second component remains bounded, too. Since the symplectic form ω is exact there are no nonconstant J -holomorphic spheres. This excludes bubbling and hence the derivatives of (4) can be controlled, see [32]. This proves the claim. We now set

$$n(c_-, c_+) = \#\mathcal{M}_{c_-, c_+}^0(\mathcal{A}^H, h, J, g_c) \bmod 2 \in \mathbb{Z}_2$$

and define the Floer boundary operator

$$\partial: CF(\mathcal{A}^H, h) \rightarrow CF(\mathcal{A}^H, h)$$

as the linear extension of

$$\partial c = \sum_{c' \in \text{crit}(h)} n(c, c') c'$$

for $c \in \text{crit}(h)$. Again using the fact that the moduli space of cascades are compact modulo breaking, a well-known argument in Floer theory shows that

$$\partial^2 = 0.$$

We define our Floer homology as usual by

$$HF(\mathcal{A}^H, h, J, g_c) = \frac{\ker \partial}{\text{im } \partial}.$$

Standard arguments show that $HF(\mathcal{A}^H, h, J, g_c)$ is independent of the choices of h , J , and g_c up to canonical isomorphism and hence $HF(\mathcal{A}^H)$ is well defined. To prove that it is invariant under homotopies of H we use Corollary 3.3 to show that the Floer homotopies which are defined by counting solutions of the s -dependent gradient equation are well defined. This finishes the proof of Theorem 1.1. \square

Proof of Theorem 1.2: We consider the following perturbation of \mathcal{A}^H . Let $F \in C^\infty(\mathbb{R}/\mathbb{Z} \times V)$ be a smooth map such that $F|_{(0,1) \times V}$ has compact support. We use the notation $F_t = F(t, \cdot)$ for $t \in \mathbb{R}/\mathbb{Z}$. Denote by ϕ_H^t and ϕ_F^t the flows of the Hamiltonian vector fields of H and F_t , respectively. We define

$$\mathcal{A}_F^H: \mathcal{L} \times \mathbb{R} \rightarrow \mathbb{R}$$

by

$$\mathcal{A}_F^H(v, \eta) := \mathcal{A}^H(v, \eta) - \int_0^1 F_t(\phi_H^{-t\eta}(v(t))) dt.$$

We further abbreviate

$$\mathfrak{S}(X_H) := \text{cl}\{x \in M : X_H(x) \neq 0\}$$

the support of the Hamiltonian vector field of H . Theorem 1.2 follows from the following Proposition and a standard Floer homotopy argument. \square

Proposition 3.5 *Assume that $\phi_F^1(\mathfrak{S}(X_H)) \cap \mathfrak{S}(X_H) = \emptyset$. Then there exists $\tilde{F} \in C^\infty(\mathbb{R}/\mathbb{Z} \times V)$ such that $\tilde{F}|_{(0,1) \times V}$ has compact support and there are no critical points of $\mathcal{A}_{\tilde{F}}^H$.*

Proof: Critical points of \mathcal{A}_F^H are solutions of the problem

$$\left. \begin{aligned} \partial_t v(t) &= \left(\phi_H^{-\eta t}\right)^* X_{F_t}(v(t)) + \eta X_H(v(t)), \quad t \in \mathbb{R}/\mathbb{Z}, \\ \int_0^1 (t\{H, F_t\}(\phi_H^{-t\eta}(v(t)) + H(v(t)))) dt &= 0 \end{aligned} \right\}$$

with the Poisson bracket given by $\{F, H\} = dF(X_H)$. Define $w \in C^\infty([0, 1], M)$ by

$$w(t) := \phi_H^{-t\eta}(v(t)), \quad t \in [0, 1].$$

Then w satisfies

$$\left. \begin{aligned} \partial_t w(t) &= X_{F_t}(w(t)), \quad t \in [0, 1], \\ w(1) &= \phi_H^{-\eta}(w(0)), \\ \int_0^1 (t\{H, F_t\}(w(t)) + H(w(t))) dt &= 0. \end{aligned} \right\} \quad (12)$$

For a smooth map $\rho \in C^\infty([0, 1], [0, 1])$ satisfying $\rho(0) = 0$ and $t \in \mathbb{R}$ set

$$F_t^\rho := \dot{\rho}(t)F_{\rho(t)} \in C^\infty(V).$$

Note that

$$\phi_{F^\rho}^t = \phi_F^{\rho(t)}.$$

Equations (12) for F replaced by F^ρ become

$$\left. \begin{aligned} w(t) &= \phi_F^{\rho(t)}(w(0)), \\ w(1) &= \phi_H^{-\eta}(w(0)), \\ \int_0^1 (t\dot{\rho}(t)\{H, F_{\rho(t)}\}(w(t)) + H(w(t))) dt &= 0. \end{aligned} \right\} \quad (13)$$

Since the Hamiltonian vector field of H has compact support, there exists a constant c such that

$$\max_{\substack{x \in V, \\ t \in \mathbb{R}/\mathbb{Z}}} |\{H, F_t\}(x)| \leq c, \quad \max_{x \in V} |H(x)| \leq c.$$

Using again that the support of the Hamiltonian vector field is compact together with the fact that 0 is a regular value of H we conclude that there exists $\delta > 0$ such that

$$\min_{x \in V \setminus \mathfrak{S}(X_H)} |H(x)| = \delta.$$

Choose an $\epsilon > 0$ such that

$$\epsilon < \frac{\delta}{2c + \delta}$$

and a smooth function $\rho_\epsilon \in C^\infty([0, 1], [0, 1])$ such that

$$\left. \begin{aligned} \rho_\epsilon(0) &= 0 \\ \rho_\epsilon(t) &= 1, \quad t \geq \epsilon, \\ \dot{\rho}_\epsilon(t) &\geq 0, \quad t \in [0, 1]. \end{aligned} \right\}$$

Proposition 3.5 follows now with $\tilde{F} = F^{\rho_\epsilon}$ in view of the lemma below. \square

Lemma 3.6 *Assume that $\phi_F^1(\mathfrak{S}(X_H)) \cap \mathfrak{S}(X_H) = \emptyset$. Then there are no solution of (13) for $\rho = \rho_\epsilon$.*

Proof: Let w be a solution of (13). We first claim that

$$w(0) \notin \mathfrak{S}(X_H). \quad (14)$$

We argue by contradiction and assume that $w(0) \in \mathfrak{S}(X_H)$. It follows from the first equation in (13) and the assumption of the lemma that

$$w(1) = \phi_F^{\rho_\epsilon(1)}(w(0)) = \phi_F^1(w(0)) \notin \mathfrak{S}(X_H).$$

The definition of $\mathfrak{S}(X_H)$ implies that

$$\phi_H^\eta(w(1)) = w(1).$$

Combining the above two equations together with the second equation in (13) we conclude

$$w(0) = \phi^\eta(w(1)) = w(1) \notin \mathfrak{S}(X_H).$$

This contradicts the assumption that $w(0) \in \mathfrak{S}(X_H)$ and proves (14). Combining (14) with the second equation in (13) we obtain

$$w(1) = w(0) \notin \mathfrak{S}(X_H). \quad (15)$$

Using the definition of ρ_ϵ , the first equation in (13), and (15) we get

$$w(t) = \phi_F^{\rho_\epsilon(t)}(w(0)) = \phi_F^1(w(0)) = w(1) \notin \mathfrak{S}(X_H), \quad t \geq \epsilon. \quad (16)$$

Using the definition of $\mathfrak{S}(X_H)$ and of δ we deduce that

$$|H(w(t))| \geq \delta, \quad \{H, F_{\rho_\epsilon(t)}\}(w(t)) = 0, \quad t \geq \epsilon. \quad (17)$$

Using (17), the definition of c and ϵ , and the properties of ρ_ϵ we estimate

$$\begin{aligned}
& \left| \int_0^1 (t\dot{\rho}_\epsilon(t)\{H, F_{\rho_\epsilon(t)}\}(w(t)) + H(w(t))) dt \right| \\
& \geq - \left| \int_0^\epsilon (t\dot{\rho}_\epsilon(t)\{H, F_{\rho_\epsilon(t)}\}(w(t)) + H(w(t))) dt \right| \\
& \quad + \left| \int_\epsilon^1 (t\dot{\rho}_\epsilon(t)\{H, F_{\rho_\epsilon(t)}\}(w(t)) + H(w(t))) dt \right| \\
& \geq - \left| \int_0^\epsilon (\epsilon c \dot{\rho}_\epsilon(t) + c) dt \right| + \delta(1 - \epsilon) \\
& = -2c\epsilon + \delta(1 - \epsilon) \\
& > 0.
\end{aligned}$$

This contradicts the third equation in (13). Hence there are no solutions of (13), which proves the lemma. \square

4 Index computations

In this section we prove Theorem 1.3. The proof comes down to the computation of the indices of generators of the Floer chain complex in the case that Σ is the unit cotangent bundle of the sphere.

We first have to study the question under which conditions $HF(\Sigma, V)$ has a \mathbb{Z} -grading. Throughout this section, we make the following assumptions:

- (A) Closed Reeb orbits on (Σ, α) are of Morse-Bott type (see Section 3).
- (B) Σ is simply connected and V satisfies $I_{c_1} = 0$.

Under these assumptions the (transversal) Conley Zehnder index of a Reeb orbit $v \in C^\infty(S^1, \Sigma)$ can be defined in the following way. Since Σ is simply connected, we can find a map $\bar{v} \in C^\infty(D, \Sigma)$ on the unit disk $D = \{z \in \mathbb{C} : |z| \leq 1\}$ such that $\bar{v}(e^{2\pi i t}) = v(t)$. Choose a (homotopically unique) symplectic trivialization of the symplectic vector bundle $(\bar{v}^*\xi, \bar{v}^*d\alpha)$. The linearized flow of the Reeb vector field along v defines a path in the group $Sp(2n - 2, \mathbb{R})$ of symplectic matrices. The Maslov index of this path [38] is the (*transversal*) *Conley-Zehnder index* $\mu_{CZ} \in \frac{1}{2}\mathbb{Z}$. It is independent of the choice of the disk \bar{v} due to the assumption $I_{c_1} = 0$ on V .

Let \mathcal{M} be the moduli space of all finite energy gradient flow lines of the action functional \mathcal{A}^H . Since \mathcal{A}^H is Morse-Bott every finite energy gradient flow line $(v, \eta) \in C^\infty(\mathbb{R} \times S^1, V) \times C^\infty(\mathbb{R}, \mathbb{R})$ converges exponentially at both ends to critical points $(v^\pm, \eta^\pm) \in \text{crit}(\mathcal{A}^H)$ as the flow parameter goes to $\pm\infty$. The

linearization of the gradient flow equation along any path (v, η) in $\mathcal{L} \times \mathbb{R}$ which converges exponentially to the critical points of \mathcal{A}^H gives rise to an operator $D_{(v, \eta)}^{\mathcal{A}^H}$. For suitable weighted Sobolev spaces (the weights are needed because we are in a Morse-Bott situation) the operator $D_{(v, \eta)}^{\mathcal{A}^H}$ is a Fredholm operator. Let $C^-, C^+ \subset \text{crit}(\mathcal{A}^H)$ be the connected components of the critical manifold of \mathcal{A}^H containing (v^-, η^-) or (v^+, η^+) respectively. The local virtual dimension of \mathcal{M} at a finite energy gradient flow line is defined to be

$$\text{vir} \dim_{(v, \eta)} \mathcal{M} := \text{ind} D_{(v, \eta)}^{\mathcal{A}^H} + \dim C^- + \dim C^+ \quad (18)$$

where $\text{ind} D_{(v, \eta)}^{\mathcal{A}^H}$ is the Fredholm index of the Fredholm operator $D_{(v, \eta)}^{\mathcal{A}^H}$. For generic compatible almost complex structures, the moduli space of finite energy gradient flow lines is a manifold and the local virtual dimension of the moduli space at a gradient flow line (v, η) corresponds to the dimension of the connected component of \mathcal{M} containing (v, η) . Our first goal is to prove the following index formula.

Proposition 4.1 *Assume that hypotheses (A) and (B) hold. Let $C^-, C^+ \subset \text{crit}(\mathcal{A}^H)$ be two connected components of the critical manifold of \mathcal{A}^H . Let $(v, \eta) \in C^\infty(\mathbb{R} \times S^1, V) \times C^\infty(\mathbb{R}, \mathbb{R})$ be a gradient flow line of \mathcal{A}^H which converges at both ends $\lim_{s \rightarrow \pm\infty} (v, \eta)(s) \rightarrow (v^\pm, \eta^\pm)$ to critical points of \mathcal{A}^H satisfying $(v^\pm, \eta^\pm) \in C^\pm$. Choose maps $\bar{v}^\pm \in C^\infty(D, \Sigma)$ satisfying $\bar{v}^\pm(e^{2\pi i t}) = v^\pm(t)$. Then the local virtual dimension of the moduli space \mathcal{M} of finite energy gradient flow lines of \mathcal{A}^H at (v, η) is given by*

$$\text{vir} \dim_{(v, \eta)} \mathcal{M} = \mu_{CZ}(v^+) - \mu_{CZ}(v^-) + 2c_1(\bar{v}^- \# v \# \bar{v}^+) + \frac{\dim C^- + \dim C^+}{2} \quad (19)$$

where $\bar{v}^- \# v \# \bar{v}^+$ is the sphere obtained by capping the cylinder v with the disks \bar{v}^+ and \bar{v}^- , and $c_1 = c_1(TV)$.

The proof is based on a discussion of

Spectral flows. It is shown in [40] that the Fredholm index of $D_{(v, \eta)}^{\mathcal{A}^H}$ can be computed via the *spectral flow* μ_{spec} (see Appendix B) of the Hessian $\text{Hess}_{\mathcal{A}^H}$ along (v, η) by the formula

$$\text{ind} D_{(v, \eta)}^{\mathcal{A}^H} = \mu_{\text{spec}} \left(\text{Hess}_{\mathcal{A}^H}(v, \eta) \right). \quad (20)$$

Our proof compares the spectral flow of the Hessian of \mathcal{A}^H with the spectral flow of the action functional of classical mechanics which can be computed via the Conley-Zehnder indices. For a *fixed* Lagrange multiplier $\eta \in \mathbb{R}$ the action functional of classical mechanics arises as

$$\mathcal{A}_\eta^H := \mathcal{A}^H(\cdot, \eta): \mathcal{L} \rightarrow \mathbb{R}.$$

Assume first that the periods η^\pm of the Reeb orbits v^\pm are nonzero. We begin by homotoping the action functional \mathcal{A}^H via Morse-Bott functionals with fixed critical manifold to an action functional \mathcal{A}^{H^1} which satisfies the assumptions of (the infinite dimensional analogue of) Lemma B.6. There exists a neighbourhood $U \subset V$ of Σ and an $\epsilon > 0$ such that U is symplectomorphic to $(\Sigma \times (-\epsilon, \epsilon), d(e^r \alpha))$ where r is the coordinate on $(-\epsilon, \epsilon)$. Since \mathring{A}^H is Morse-Bott and the Hamiltonian vector field $X_H(x)$ for $x \in \Sigma$ equals the Reeb vector field $R(x)$, there exists a homotopy H^s for $s \in [0, 1]$ which satisfies the following conditions:

- $H^0 = H$.
- $X_{H^s}(x) = R(x)$ for $x \in \Sigma$ and $s \in [0, 1]$.
- There exist neighbourhoods $U^\pm \subset U$ of the critical manifolds C^\pm and functions $h_\pm \in C^\infty((-\epsilon, \epsilon))$ satisfying $h_\pm(0) = 0$, $h'_\pm(0) = 1$, $h''_\pm(0) \neq 0$, and $h'_\pm(r) \neq 0$ for $r \in (-\epsilon, \epsilon)$ such that $H^1(x, r) = h_\pm(r)$ for $(x, r) \in U^\pm \subset \Sigma \times (-\epsilon, \epsilon)$.
- \mathring{A}^{H^s} is Morse-Bott for all $s \in [0, 1]$.

Here the signs of $h''_\pm(0)$ are determined by the second derivatives of H in the direction transverse to Σ along C^\pm . Since \mathcal{A}^H can be homotoped to \mathcal{A}^{H^1} via Morse-Bott action functionals with fixed critical manifold, we obtain

$$\mu_{\text{spec}}\left(\text{Hess}_{\mathcal{A}^H}(v, \eta)\right) = \mu_{\text{spec}}\left(\text{Hess}_{\mathcal{A}^{H^1}}(v, \eta)\right). \quad (21)$$

If $(v_0, \eta_0) \in C^\infty(S^1, \Sigma \cap U^\pm) \times \mathbb{R}$ is a critical point of \mathcal{A}^H , then (v_0, η_0) is also a critical point of \mathcal{A}^{H^1} . Moreover, the family $(v_\rho, \eta_\rho) \in C^\infty(S^1, U) \times \mathbb{R}$ given by

$$v_\rho(t) = (v_0(t), h^{-1}(-\rho)), \quad \eta_\rho = \frac{\eta_0}{h'(h^{-1}(-\rho))}$$

consists of critical points for the family of action functionals $\mathcal{A}^{H^1, \rho}: \mathcal{L} \times \mathbb{R} \rightarrow \mathbb{R}$ given for $(v, \eta) \in \mathcal{L} \times \mathbb{R}$ by

$$\mathcal{A}^{H^1, \rho}(v, \eta) := \int v^* \lambda - \eta \left(\int_0^1 H^1(v(t)) dt + \rho \right).$$

Note that

$$\partial_\rho \eta_\rho|_{\rho=0} = -\frac{\eta_0 h''_\pm(0)}{h'_\pm(0)^2}.$$

Hence for $\eta_0 = \eta^\pm \neq 0$ the hypotheses of Lemma B.6 are satisfied. It follows from Theorem B.5 and Lemma B.6 that the spectral flow can be expressed in terms of the spectral flow of the action functional of classical mechanics plus a correction term accounting for the second derivatives of H transversally to Σ as

$$\mu_{\text{spec}}\left(\text{Hess}_{\mathcal{A}^{H^1}}(v, \eta)\right) = \mu_{\text{spec}}\left(\text{Hess}_{\mathcal{A}^{H^1}}(v)\right) + \frac{1}{2} \left(\text{sign}(\eta^- \cdot h''_-(0)) - \text{sign}(\eta^+ \cdot h''_+(0)) \right). \quad (22)$$

It follows from a theorem due to Salamon and Zehnder [43] that the spectral flow of the Hessian of $\mathcal{A}_\eta^{H^1}$ can be computed via Conley-Zehnder indices. However, the Conley-Zehnder indices in the Salamon-Zehnder theorem are not the (transversal) Conley-Zehnder indices explained above, but the Maslov index of the linearized flow of the Reeb vector field on the whole tangent space of V and not just on the contact hyperplane. For a Reeb orbit v we will denote this second (full) Conley-Zehnder index by $\hat{\mu}_{CZ}(v)$. Note that $\hat{\mu}_{CZ}(v)$ depends on the second derivatives of H transversally to Σ while $\mu_{CZ}(v)$ does not. Another complication is that we are in a Morse-Bott situation and we have to adapt the Salamon-Zehnder theorem to this situation. Formula (35) defines the spectral flow also for Morse-Bott situations. To adopt the Conley-Zehnder indices to the Morse-Bott situation observe that in a symplectic trivialization the linearized flow of the Reeb vector field can be expressed as a solution of an ordinary differential equation

$$\dot{\Psi}(t) = J_0 S(t) \Psi(t), \quad \Psi(0) = \text{id},$$

where $t \mapsto S(t) = S(t)^T$ is a smooth path of symmetric matrices. For a real number δ we define Ψ_δ as the solution of

$$\dot{\Psi}_\delta(t) = J_0(S(t) - \delta \cdot \text{id}) \Psi_\delta(t), \quad \Psi_\delta(0) = \text{id},$$

and set $\mu_{CZ}^\delta(v)$, respectively $\hat{\mu}_{CZ}^\delta(v)$ as the Conley-Zehnder index of Ψ_δ where in the first case we restrict Ψ_δ to the contact hyperplane and in the second case we consider it on the whole tangent space. We put

$$\mu_{CZ}^+(v) := \lim_{\delta \searrow 0} \mu_{CZ}^\delta(v), \quad \mu_{CZ}^-(v) := \lim_{\delta \searrow 0} \mu_{CZ}^{-\delta}(v)$$

and analogously $\hat{\mu}_{CZ}^+(v)$ and $\hat{\mu}_{CZ}^-(v)$. Note that while $\hat{\mu}_{CZ}(v)$ and $\mu_{CZ}(v)$ are half-integers, $\hat{\mu}_{CZ}^\omega(v)$ and $\mu_{CZ}^\pm(v)$ are actually integers. We are now in position to state the theorem of Salamon and Zehnder.

Theorem 4.2 (Salamon-Zehnder [43]) *The spectral flow of the Hessian of $\mathcal{A}_\eta^{H^1}$ is given by*

$$\mu(H_{\mathcal{A}_\eta^{H^1}}(v)) = \hat{\mu}_{CZ}^+(v^+) - \hat{\mu}_{CZ}^-(v^-) + 2c_1(\bar{v}^- \# v \# \bar{v}^+).$$

Relations between Conley-Zehnder indices. The following two lemmata relate the different Conley-Zehnder indices to each other.

Lemma 4.3 *For a Reeb orbit v with period $\eta \neq 0$, viewed as a 1-periodic orbit of the Hamiltonian vector field of ηH , we have*

$$\hat{\mu}_{CZ}^\pm(v) = \mu_{CZ}^\pm(v) + \frac{1}{2} \left(\text{sign}(\eta h''(0)) \mp 1 \right).$$

Proof: By the product property [42] of the Conley-Zehnder index the difference of $\hat{\mu}_{CZ}^{\pm}(v)$ and $\mu_{CZ}^{\pm}(v)$ is given by the Conley-Zehnder index of the linearized flow of the Hamiltonian vector field restricted to the symplectic orthogonal complement ξ^{ω} of the contact hyperplane in the tangent space of V . With respect to the trivialization $\mathbb{C} \rightarrow \xi^{\omega}(v(t))$ given by $x + iy \mapsto (x \cdot \nabla H(v(t)) + y \cdot X_H(v(t)))$ for $t \in S^1$, the linearized flow of the Hamiltonian vector field is given by

$$\Psi(t) = \begin{pmatrix} 1 & 0 \\ t\eta h''(0) & 1 \end{pmatrix}.$$

Abbreviate $a := \eta \cdot h''(0)$. A computation shows that

$$\Psi_{\delta}(t) = e^{\delta(a-\delta)t^2} \begin{pmatrix} 1 & \delta t \\ (a-\delta)t & 1 \end{pmatrix}.$$

Recall [42] that the Conley-Zehnder index can be computed in terms of crossing numbers, where a number $t \in [0, 1]$ is called a crossing if $\det(\text{id} - \Psi_{\delta}(t)) = 0$. The formula above shows that for δ small enough the only crossing happens at zero. Hence by [42] the Conley-Zehnder index is given by

$$\mu_{CZ}(\Psi_{\delta}) = \frac{1}{2} \text{sign} \begin{pmatrix} a - \delta & 0 \\ 0 & -\delta \end{pmatrix}.$$

If $|\delta| < |a|$ we obtain

$$\mu_{CZ}(\Psi_{\delta}) = \frac{1}{2} \left(\text{sign}(a) - \text{sign}(\delta) \right) = \frac{1}{2} \left(\text{sign}(\eta h''(0)) - \text{sign}(\delta) \right)$$

and hence

$$\hat{\mu}_{CZ}^{\pm}(v) - \mu_{CZ}^{\pm}(v) = \frac{1}{2} \left(\text{sign}(\eta h''(0)) \mp 1 \right).$$

This proves the lemma. \square

Lemma 4.4 *Let v be a Reeb orbit with period $\eta \neq 0$ and C_v the component of the critical manifold of \mathcal{A}^{H^1} which contains v . Then*

$$\hat{\mu}_{CZ}(v) = \hat{\mu}_{CZ}^{\pm}(v) \pm \frac{\dim C_v}{2}, \quad \mu_{CZ}(v) = \mu_{CZ}^{\pm}(v) \pm \frac{\dim C_v - 1}{2}.$$

Proof: Obviously

$$\hat{\mu}_{CZ}^{-}(v) - \hat{\mu}_{CZ}^{+}(v) = \dim C_v, \quad \mu_{CZ}^{-}(v) - \mu_{CZ}^{+}(v) = \dim C_v - 1. \quad (23)$$

The reason for the minus one in the second formula is that the transversal Conley-Zehnder index only takes into account the critical manifold of \mathcal{A}^{H^1} modulo the S^1 -action given by the Reeb vector field. The Conley-Zehnder index can be interpreted as intersection number of a path of Lagrangian subspaces with the Maslov cycle, see [38]. Under a small perturbation the intersection number

can only change at the initial and endpoint. Since the Lagrangian subspace at the initial point is fixed it will change only at the endpoint. There the contribution is given by half of the crossing number which equals $\dim C_v$ in the case one considers the Conley-Zehnder index on the whole tangent space respectively $\dim C_v - 1$ if one considers the Conley-Zehnder index only on the contact hyperplane. In particular,

$$|\hat{\mu}_{CZ}(v) - \hat{\mu}_{CZ}^\pm(v)| \leq \frac{\dim C_v}{2}, \quad |\mu_{CZ}(v) - \mu_{CZ}^\pm(v)| \leq \frac{\dim C_v - 1}{2}. \quad (24)$$

Comparing (23) and (24) the lemma follows. \square

Proof of Proposition 4.1: We first assume that η^- and η^+ are nonzero. Combining the theorem of Salamon and Zehnder (Theorem 4.2) with Lemma 4.3 and Lemma 4.4 we obtain

$$\begin{aligned} \mu(H_{\mathcal{A}_\eta^{H^1}}(v)) &= \hat{\mu}_{CZ}^+(v^+) - \hat{\mu}_{CZ}^-(v^-) + 2c_1(\bar{v}^- \# v \# \bar{v}^+) \\ &= \mu_{CZ}^+(v^+) - \mu_{CZ}^-(v^-) + 2c_1(\bar{v}^- \# v \# \bar{v}^+) - 1 \\ &\quad + \frac{1}{2} \left(\text{sign}(\eta^+ \cdot h_+''(0)) - \text{sign}(\eta^- \cdot h_-''(0)) \right) \\ &= \mu_{CZ}(v^+) - \mu_{CZ}(v^-) + 2c_1(\bar{v}^- \# v \# \bar{v}^+) - \frac{\dim C^- + \dim C^+}{2} \\ &\quad + \frac{1}{2} \left(\text{sign}(\eta^+ \cdot h_+''(0)) - \text{sign}(\eta^- \cdot h_-''(0)) \right). \end{aligned}$$

Combining this equality with (18), (20), (21), and (22) we compute

$$\begin{aligned} \text{virdim}_{(v,\eta)} \mathcal{M} &= \text{ind} D_{(v,\eta)}^{\mathcal{A}^H} + \dim C^- + \dim C^+ \\ &= \mu(H_{\mathcal{A}^H}(v, \eta)) + \dim C^- + \dim C^+ \\ &= \mu(H_{\mathcal{A}^{H^1}}(v, \eta)) + \dim C^- + \dim C^+ \\ &= \mu(H_{\mathcal{A}_\eta^{H^1}}(v)) + \frac{1}{2} \left(\text{sign}(\eta^- \cdot h_-''(0)) - \text{sign}(\eta^+ \cdot h_+''(0)) \right) \\ &\quad + \dim C^- + \dim C^+ \\ &= \mu_{CZ}(v^+) - \mu_{CZ}(v^-) + 2c_1(\bar{v}^- \# v \# \bar{v}^+) \\ &\quad + \frac{\dim(C^-) + \dim(C^+)}{2}. \end{aligned}$$

This proves the proposition for the case where the periods of the asymptotic Reeb orbits are both nonzero. To treat also the case where one of the asymptotic Reeb orbits is constant we consider the following involution on the loop space \mathcal{L}

$$I(v)(t) = v(-t), \quad v \in \mathcal{L}, \quad t \in S^1.$$

We extend this involution to an involution on $\mathcal{L} \times \mathbb{R}$ which we denote by abuse of notation also by I and which is given by

$$I(v, \eta) = (I(v), -\eta), \quad (v, \eta) \in \mathcal{L} \times \mathbb{R}.$$

The action functional \mathcal{A}^H transforms under the involution I by

$$\mathcal{A}^H(I(v, \eta)) = -\mathcal{A}^H(v, \eta), \quad (v, \eta) \in \mathcal{L} \times \mathbb{R}.$$

In particular, the restriction of the involution I to the critical manifold of \mathcal{A}^H induces an involution on $\text{crit}(\mathcal{A}^H)$ and the fixed points of this involution are the constant Reeb orbits.

We consider now a finite energy gradient flow line $(v, \eta) \in C^\infty(\mathbb{R} \times S^1, V) \times C^\infty(\mathbb{R}, \mathbb{R})$ of the action functional \mathcal{A}^H whose right end (v^+, η^+) is a constant Reeb orbit and whose left end (v^-, η^-) is a nonconstant Reeb orbit. For the path (v, η) in $\mathcal{L} \times \mathbb{R}$ we consider the path $(v, \eta)_I = (v_I, \eta_I)$ in $\mathcal{L} \times \mathbb{R}$ defined by $(v, \eta)_I(s) = I(v, \eta)(-s)$ for $s \in \mathbb{R}$. The path $(v, \eta)_I$ goes from (v^+, η^+) to $I(v^-, \eta^-)$ and gluing the paths (v, η) and $(v, \eta)_I$ together we obtain a path $(v, \eta) \# (v, \eta)_I$ from (v^-, η^-) to $I(v^-, \eta^-)$. The Fredholm indices of the different paths are related by

$$\text{ind} D_{(v, \eta)}^{\mathcal{A}^H} = \text{ind} D_{(v, \eta)_I}^{\mathcal{A}^H}, \quad \text{ind} D_{(v, \eta) \# (v, \eta)_I}^{\mathcal{A}^H} = \text{ind} D_{(v, \eta)}^{\mathcal{A}^H} + \text{ind} D_{(v, \eta)_I}^{\mathcal{A}^H} + \dim C^+.$$

From this we compute, using (19) for the case of nonconstant Reeb orbits and the equality $\mu_{CZ}(I(v^\pm)) = -\mu_{CZ}(v^\pm)$,

$$\begin{aligned} \text{ind} D_{(v, \eta)}^{\mathcal{A}^H} &= \frac{1}{2} \cdot \text{ind} D_{(v, \eta) \# (v, \eta)_I}^{\mathcal{A}^H} - \frac{\dim C^+}{2} \\ &= \frac{1}{2} \left(\mu_{CZ}(I(v^-)) - \mu_{CZ}(v^-) + 2c_1(\bar{v}^- \# v \# v_I \# I\bar{v}^+) \right. \\ &\quad \left. - \frac{\dim C^- + \dim IC^-}{2} \right) - \frac{\dim C^+}{2} \\ &= -\mu_{CZ}(v^-) + 2c_1(\bar{v}^- \# v) - \frac{\dim C^- + \dim C^+}{2}, \end{aligned}$$

from which we deduce (19) using (18). This proves the proposition for the case of gradient flow lines whose left end is a constant Reeb orbit. The case of gradient flow lines whose right end is constant can be deduced in the same way or by considering the coindex. This finishes the proof of the Proposition 4.1. \square

In order to define a \mathbb{Z} -grading on $HF(\Sigma, V)$ we need that the local virtual dimension just depends on the asymptotics of the finite energy gradient flow line. By (19) this is the case if $I_{c_1} = 0$ on V . In this case the local virtual dimension is given by

$$\text{virdim}_{(v, \eta)} \mathcal{M} = \mu_{CZ}(v^+) - \mu_{CZ}(v^-) + \frac{\dim C^- + \dim C^+}{2}. \quad (25)$$

In order to deal with the third term it is useful to introduce the following index for the Morse function h on $\text{crit}(\mathcal{A}^H)$. We define the *signature index* $\text{ind}_h^\sigma(c)$ of a critical point c of h to be

$$\text{ind}_h^\sigma(c) := -\frac{1}{2} \text{sign}(\text{Hess}_h(c)),$$

see Appendix A. The half signature index is related to the *Morse index* $\text{ind}_h^m(c)$, given by the number of negative eigenvalues of $\text{Hess}_h(c)$ counted with multiplicity, by

$$\text{ind}_h^\sigma(c) = -\text{ind}_h^m(c) - \frac{1}{2}\dim_c(\text{crit}(\mathcal{A}^H)). \quad (26)$$

We define a *grading* μ on $CF_*(\mathcal{A}^H, h)$ by

$$\mu(c) := \mu_{CZ}(c) + \text{ind}_h^\sigma(c).$$

By considering the case of nondegenerate closed Reeb orbits, one sees that μ takes values in the set $\frac{1}{2} + \mathbb{Z}$, so it is indeed a \mathbb{Z} -grading (shifted by $\frac{1}{2}$). Using equation (25), it is shown in Appendix A that the Floer boundary operator ∂ has degree -1 with respect to this grading. Hence we get a \mathbb{Z} -grading on the homology $HF_*(\Sigma, V)$.

Proof of Theorem 1.3: To prove Theorem 1.3 we use the fact that the chain groups underlying the Floer homology $HF_*(\mathcal{A}^H)$ only depend on (Σ, α) and not on the embedding of Σ into V . We show that for the unit cotangent bundle S^*S^n for $n \geq 4$ the Floer homology equals the chain complex. More precisely, we choose the standard round metric on S^n normalized such that all geodesics are closed with minimal period one. For this choice assumption (A) is satisfied. The critical manifold of \mathcal{A}^H consists of \mathbb{Z} copies of S^*S^n , where \mathbb{Z} corresponds to the period of the geodesic. There is a Morse function h_0 on S^*S^n with precisely 4 critical points and zero boundary operator (with \mathbb{Z}_2 -coefficients!) whose Morse homology satisfies

$$HM_k(S^*S^n; \mathbb{Z}_2) = CM_k(h_0; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & k \in \{0, n-1, n, 2n-1\} \\ 0 & \text{else.} \end{cases}$$

Let h be the Morse function on the critical manifold which coincides with h_0 on each connected component. The chain complex is generated by

$$\text{crit}(h) \cong \mathbb{Z} \times \text{crit}(h_0).$$

A closed geodesic c is also a critical point of the energy functional on the loop space. The *index* $\text{ind}_E(c)$ of a closed geodesic is defined to be the Morse index of the energy functional at the geodesic and the *nullity* $\nu(c)$ is defined to be the dimension of the connected component of the critical manifold of the energy functional which contains the geodesic minus one. The (transverse) Conley-Zehnder index of a closed geodesic is given by

$$\mu_{CZ}(c) = \text{ind}_E(c) + \frac{\nu(c)}{2}. \quad (27)$$

This is proved in [10, 47] for nondegenerate geodesics; the degenerate case follows from the nondegenerate one using the averaging property of the Conley-Zehnder index (Lemma 4.4). By the Morse index theorem, see [35] or [28, Theorem 2.5.14], the index of a geodesic is given by the number of conjugate

points counted with multiplicity plus the concavity. The latter one vanishes for the standard round metric on S^n , since each closed geodesic has a variation of closed geodesics having the same length [49].

Using the Morse index theorem and equations (27) and (26), we compute the index of $(m, x) \in \mathbb{Z} \times \text{crit}(h_0)$:

$$\begin{aligned} \mu(m, x) &= \mu_{CZ}(m, x) + \text{ind}_h^\sigma(x) \\ &= \text{ind}_E(m, x) + \frac{\nu(m, x)}{2} + \text{ind}_h^\sigma(x) \\ &= (2m - 1)(n - 1) + \frac{2n - 2}{2} + \text{ind}_h^\sigma(x) \\ &= 2m(n - 1) + \text{ind}_h^m(x) - \frac{2n - 1}{2}. \end{aligned}$$

It follows from Lemma 3.2 that the action satisfies

$$\mathcal{A}^H(m, x) = m.$$

In order to have a gradient flow line of \mathcal{A}^H from a critical point (m_1, x_1) to a critical point (m_2, x_2) we need

$$\mathcal{A}^H(m_2, x_2) - \mathcal{A}^H(m_1, x_1) = m_2 - m_1 > 0$$

and

$$\mu(m_2, x_2) - \mu(m_1, x_1) = 2(m_2 - m_1)(n - 1) + (i_2 - i_1) = 1$$

for $i_1, i_2 \in \{0, n - 1, n, 2n - 1\}$, which is impossible if $n \geq 4$. Hence there are no gradient flow lines, so the Floer homology equals the chain complex. This proves Theorem 1.3. \square

A Morse-Bott homology

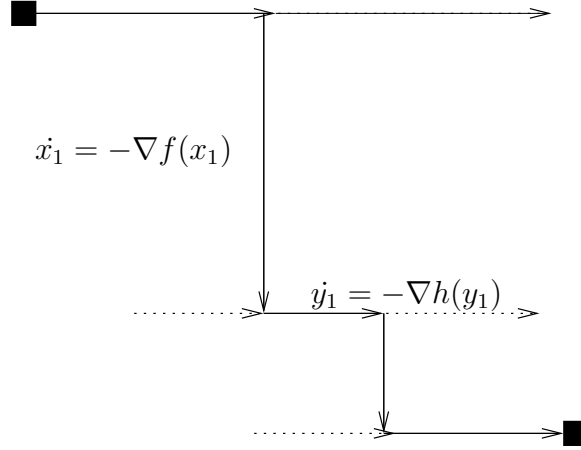
We briefly indicate in this appendix how to define Morse-Bott homology by the use of gradient flow lines with cascades. More details of this approach can be found in [19, Appendix A]. We begin with the finite dimensional situation. Let M be a manifold and $f \in C^\infty(M)$ a *Morse-Bott function*, i.e. the critical set $\text{crit}(f)$ is a manifold and

$$T_x \text{crit}(f) = \ker \text{Hess}_f(x), \quad x \in \text{crit}(f),$$

where $\text{Hess}_f(x)$ denotes the Hessian of f at x . We then choose an additional Morse function h on $\text{crit}(f)$. The chain group for Morse-Bott homology is the \mathbb{Z}_2 -vector space given by

$$CM(f, h) := \text{crit}(h) \otimes \mathbb{Z}_2.$$

Morse-Bott homology should also be definable over the integers via the cascade approach, but this is nowhere written down. The boundary operator is defined by counting gradient flow lines with cascades between two critical points of h which are indicated by the following picture.



A flow line with cascades

A *gradient flow line with cascades* starts with a gradient flow line of h on $\text{crit}(f)$ which converges at its negative asymptotic end to a critical point of h . In finite time this gradient flow line meets the asymptotic end of a gradient flow line of the Morse-Bott function f . We refer to this gradient flow line of f as the first cascade. The cascade converges at its positive end again to a point in $\text{crit}(f)$. There the flow continues with the gradient flow of h on $\text{crit}(f)$. After finite time a second cascade might appear but having passed through finitely many cascades we finally end up with a gradient flow line of h which we follow until it converges asymptotically to a critical point of h . Gradient flow lines with zero cascades are also allowed. They correspond to ordinary Morse flow lines of the gradient of h on the manifold $\text{crit}(f)$. For a formal definition of gradient flow lines with cascades we refer to [19].

We finally discuss the grading for Morse-Bott homology. If f is Morse then the following two index conventions are often used. Either Morse homology is graded by the *Morse index* ind_f^m , given by the dimensions of the negative eigenspaces, or by the *signature index*

$$\text{ind}_f^\sigma(x) := -\frac{1}{2}\text{signHess}_f(x), \quad x \in \text{crit}(f),$$

where sign denotes the signature of the quadratic form given by the difference of the dimensions of the positive and negative eigenspaces. The two indices are related by the following global shift

$$\text{ind}_f^\sigma = \text{ind}_f^m - \frac{1}{2}\dim(M). \quad (28)$$

In particular, if M is even dimensional then the signature index is integer valued and if M is odd dimensional then it is half integer valued. The signature index

plays an important role in Floer's semi-infinite dimensional Morse theory. There the stable and unstable manifolds are both infinite dimensional and hence the Morse index is infinite. The grading given is given by the Maslov index which can be interpreted as a signature index as explained in [39, 41].

Both the Morse index and the signature index can be defined in the same way also for a Morse-Bott function f . The corresponding indices for a pair (f, h) consisting of a Morse-Bott function f and a Morse function h on $\text{crit}(f)$ are defined by taking sums

$$\text{ind}_{f,h}^m(x) := \text{ind}_f^m(x) + \text{ind}_h^m(x), \quad \text{ind}_{f,h}^\sigma(x) := \text{ind}_f^\sigma(x) + \text{ind}_h^\sigma(x), \quad x \in \text{crit}(h).$$

The shift formula (28) continues to hold for these indices in Morse-Bott theory.

Consider now gradient flow lines with k cascades between components C_0, \dots, C_k of $\text{crit}(f)$, starting at a critical point x^+ of h on $C^+ = C_k$ and ending at a critical point x^- of h on $C^- = C_0$. For generic metric, their moduli space (divided by the \mathbb{R} -actions on the cascades) $\mathcal{M}(x^-, x^+; C_0, \dots, C_k)$ is a manifold of dimension

$$\begin{aligned} \dim \mathcal{M}(x^-, x^+; C_0, \dots, C_k) &= \text{ind}_h^m(x^+) - \text{ind}_h^m(x^-) - 1 \\ &\quad + \sum_{i=1}^k \left(\dim \mathcal{M}(C_{i-1}, C_i) - \dim C_i \right), \end{aligned}$$

where $\mathcal{M}(C_{i-1}, C_i)$ is the moduli space of gradient flow lines of f from C_{i-1} to C_i (not divided by the \mathbb{R} -action). From

$$\dim \mathcal{M}(C_{i-1}, C_i) = \text{ind}_f^m(C_i) - \text{ind}_f^m(C_{i-1}) + \dim C_i \quad (29)$$

we obtain the dimension formula in terms of Morse indices

$$\begin{aligned} \dim \mathcal{M}(x^-, x^+; C_0, \dots, C_k) &= \text{ind}_h^m(x^+) - \text{ind}_h^m(x^-) - 1 + \text{ind}_f^m(C^+) - \text{ind}_f^m(C^-) \\ &= \text{ind}_{f,h}^m(x^+) - \text{ind}_{f,h}^m(x^-) - 1. \end{aligned} \quad (30)$$

On the other hand, in the Morse-Bott case the Morse and signature indices of a critical component C are related by

$$\text{ind}_f^\sigma(C) = \text{ind}_f^m(C) - \frac{1}{2}(\dim M - \dim C).$$

Inserting this in equation (29) yields

$$\dim \mathcal{M}(C_{i-1}, C_i) = \text{ind}_f^\sigma(C_i) - \text{ind}_f^\sigma(C_{i-1}) + \frac{\dim C_i + \dim C_{i-1}}{2}, \quad (31)$$

which in turn yields the dimension formula in terms of signature indices

$$\begin{aligned} \dim \mathcal{M}(x^-, x^+; C_0, \dots, C_k) &= \text{ind}_h^m(x^+) - \text{ind}_h^m(x^-) - 1 + \text{ind}_f^\sigma(C^+) - \text{ind}_f^\sigma(C^-) \\ &\quad - \frac{\dim C^+}{2} + \frac{\dim C^-}{2} \\ &= \text{ind}_{f,h}^\sigma(x^+) - \text{ind}_{f,h}^\sigma(x^-) - 1. \end{aligned} \quad (32)$$

So we get the same formula for $\dim \mathcal{M}(x^-, x^+; C_0, \dots, C_k)$ using either Morse indices or signature indices. Since this dimension equals zero for the moduli spaces contributing to the boundary operator in Morse-Bott homology, this shows that the boundary operator has degree -1 with respect to either grading. However, we would like to point out that a mixture of Morse indices and signature indices does not lead in general to a grading on Morse-Bott homology unless all the connected components of the critical manifold of f have the same dimension.

Finally, consider the situation in Floer homology where the ambient space is infinite-dimensional, but the components of $\text{crit}(f)$ and the moduli spaces $\mathcal{M}(C_{i-1}, C_i)$ are still finite dimensional. Moreover, (under suitable hypotheses) the dimension of these moduli spaces can be expressed by a formula analogous to (31) in terms of Conley-Zehnder indices:

$$\dim \mathcal{M}(C_{i-1}, C_i) = \mu_{CZ}(C_i) - \mu_{CZ}(C_{i-1}) + \frac{\dim C_i + \dim C_{i-1}}{2}.$$

See e.g. equation (25) for the Floer homology considered in this paper. This suggests that the Conley-Zehnder index should be viewed as a signature index, and the same computation as in the finite dimensional case above yields the dimension formula

$$\dim \mathcal{M}(x^-, x^+; C_0, \dots, C_k) = \mu(x^+) - \mu(x^-) - 1 \quad (33)$$

with respect to the signature index

$$\mu(x) := \mu_{CZ}(x) + \text{ind}_h^\sigma(x).$$

Thus the boundary operator in Floer homology has degree -1 with respect to μ and μ descends to an integer grading on Floer homology. Actually, in the case considered in this paper this grading takes values in $\frac{1}{2} + \mathbb{Z}$, where the shift by $\frac{1}{2}$ reflects the 1-dimensional constraint imposed on the free loop space.

B Spectral flow

We compare in this appendix two spectral flows which appear in Lagrange multiplier type problems. To motivate this we first consider the Lagrange multiplier functional in finite dimensions. Let (M, g) be a Riemannian manifold and $(V, \langle \cdot, \cdot \rangle)$ be a Euclidean vector space. For functions $f \in C^\infty(M)$ and $h \in C^\infty(M, V)$ the *Lagrange multiplier functional* $F \in C^\infty(M \times V)$ is given by

$$F(x, v) = f(x) + \langle v, h(x) \rangle.$$

For $v \in V$ we denote by $F_v \in C^\infty(M)$ the function given by

$$F_v = F(\cdot, v).$$

The Hessian of F with respect to the metric $g \oplus g_V$ on $M \times V$, where $g_V = \langle \cdot, \cdot \rangle$ is the Euclidean scalar product on V , is given by

$$\text{Hess}_F(x, v) = \begin{pmatrix} \text{Hess}_{F_v}(x) & dh(x)^* \\ dh(x) & 0 \end{pmatrix}.$$

Here the adjoint of $dh(x)$ is taken with respect to the inner products on $T_x M$ and $T_{h(x)} V \cong V$ given by the metric g and by $\langle \cdot, \cdot \rangle$.

We compare in this appendix the spectral flows of Hess_F and Hess_{F_v} for Lagrange multiplier functionals not necessarily defined on a finite dimensional manifold. We will apply this in the following way. For $F = \mathcal{A}^H$ the functional F_v is the action functional of classical mechanics whose spectral flow can be computed via the Conley-Zehnder indices [40].

To formulate our theorem we use the set-up of Robbin and Salamon [40]. Let W and H be separable real Hilbert spaces such that $W \subset H$ is dense and the inclusion is compact. Let $A : W \rightarrow H$ be a bounded linear operator (with respect to the norms on W and H). Viewing A as an unbounded operator on H with domain $\text{dom}(A) = W$, recall the following definitions (see e.g. [27]). The adjoint operator

$$A^* : \text{dom}(A^*) := \{v \in H \mid |\langle v, Aw \rangle_H| \leq C|w|_H \text{ for all } w \in W\} \rightarrow H$$

is defined by the equation

$$\langle A^*v, w \rangle_H = \langle v, Aw \rangle_H.$$

A is called symmetric if $W \subset \text{dom}(A^*)$ and $A^*|_W = A$, i.e. $\langle A^*v, w \rangle_H = \langle v, Aw \rangle_H$ for all $v, w \in W$. A is called self-adjoint if it is symmetric and $\text{dom}(A^*) = W$. The spectrum of A is the set of all complex numbers λ such that the operator

$$A - \lambda \cdot \text{id} : W \rightarrow H$$

is not bijective. Denote by $\ker(A)$ and $\mathcal{R}(A)$ the kernel and range (= image) of A , respectively. Denote by $\mathcal{L}(W, H)$ the space of bounded linear operators and by

$$\mathcal{S}(W, H) \subset \mathcal{L}(W, H)$$

the subspace of self-adjoint operators. The following lemma clarifies these concepts.

Lemma B.1 *Let $W \subset H$ be Hilbert spaces such that the inclusion is compact and let $A : W \rightarrow H$ be a symmetric bounded linear operator. Then the following are equivalent:*

- (i) *There exists $\lambda \in \mathbb{R}$ such that $A - \lambda \cdot \text{id} : W \rightarrow H$ is bijective.*
- (ii) *A is self-adjoint considered as an unbounded operator on H with domain $\text{dom}(A) = W$.*

- (iii) One of the defect indices $d^\pm(A) := \text{codim}(\mathcal{R}(A^\mathbb{C} \pm i \cdot \text{id}), H^\mathbb{C})$ is zero. Here $A^\mathbb{C} : W^\mathbb{C} \rightarrow H^\mathbb{C}$ denotes the complex linear extension of A to the complexified Hilbert spaces.
- (iv) The spectrum of A is discrete and consists of real eigenvalues of finite multiplicity.

Proof: We first show that (i) \Rightarrow (ii), i.e. $\text{dom}(A^*) = \text{dom}(A) = W$. To see this let $v \in \text{dom}(A^*)$. Since $A - \lambda \cdot \text{id}$ is bijective and A is symmetric, there exists $w \in W$ such that

$$(A^* - \lambda \cdot \text{id})v = (A - \lambda \cdot \text{id})w = (A^* - \lambda \cdot \text{id})w,$$

which implies

$$(A - \lambda \cdot \text{id})^*(v - w) = 0.$$

Again using the fact that $A - \lambda \cdot \text{id}$ is bijective, we conclude that $(A - \lambda \cdot \text{id})^*$ is bijective and hence

$$v = w \in W.$$

It follows that A is self-adjoint with $\text{dom}(A) = W$.

If A is self-adjoint, then both defect indices are zero, see for example [27, Theorem V.3.16], so that (ii) \Rightarrow (iii).

We show that (iii) \Rightarrow (iv). Assume that $d^-(A)$ is zero, i.e. $A^\mathbb{C} - i \cdot \text{id} : W^\mathbb{C} \rightarrow H^\mathbb{C}$ is invertible. Since the inclusion $\iota : W \rightarrow H$ is compact, the operator

$$R := \iota \circ (A - i \cdot \text{id})^{-1} : H^\mathbb{C} \rightarrow H^\mathbb{C}$$

is compact. In particular, its spectrum $\sigma(R)$ consists of eigenvalues, the only accumulation point in $\sigma(R)$ is zero, and the eigenspace for each eigenvalue except zero is finite dimensional. Let $\zeta \in \mathbb{C} \setminus \{i\}$. Then the following relations hold for the ranges

$$\mathcal{R}(A^\mathbb{C} - \zeta \cdot \text{id}) = R^{-1} \mathcal{R}(R - \frac{1}{\zeta - i} \cdot \text{id})$$

and the kernels

$$\ker(A^\mathbb{C} - \zeta \cdot \text{id}) = \ker(R - \frac{1}{\zeta - i} \cdot \text{id}).$$

In particular, we have a bijection

$$\sigma(R) \setminus \{0\} \rightarrow \sigma(A^\mathbb{C}), \quad \mu \mapsto \frac{1}{\mu} + i$$

between the spectra under which the corresponding eigenspaces do not change, i.e. for every $\mu \in \sigma(R) \setminus \{0\}$ the eigenspaces satisfy

$$E_\mu(R) = E_{\frac{1}{\mu} + i}(A^\mathbb{C}) \subset H.$$

We conclude that the spectrum of A consists of discrete eigenvalues of finite multiplicity, which are real because A is symmetric. A similar argument holds

for the case that $d^+(A)$ is zero. This shows that (iii) implies (iv).
That (iv) \Rightarrow (i) is obvious. This finishes the proof of the lemma. \square

Assume in addition that V is a finite dimensional Hilbert space. Let $A \in \mathcal{S}(W, H)$ be as before and $B \in \mathcal{L}(V, H)$ be a bounded linear operator. We denote by $A_B: W \oplus V \rightarrow H \oplus V$ the bounded symmetric operator defined by

$$A_B(w, v) = (Aw + Bv, B^*w).$$

As a consequence of Lemma B.1 we get the following corollary.

Corollary B.2 *The operator A_B is in $\mathcal{S}(W \oplus V, H \oplus V)$.*

Proof: By Lemma B.1 we have to show that A_B is self-adjoint. This is true if $B = 0$. For arbitrary B this follows from Theorem V.4.3 in [27]. \square

In the following orthogonality is always understood with respect to the inner product of H and never with respect to the inner product of W .

Definition B.3 *Let $A \in \mathcal{S}(W, H)$, and $B \in \mathcal{L}(V, H)$. We say that the tuple (A, B) is regular if*

- (i) B is injective.
- (ii) A maps $\mathcal{R}(B)$ to itself and the restriction $\hat{A} := A|_{\mathcal{R}(B)}$ is bijective.

A regular pair (A, B) gives rise to the symmetric form

$$S_{A,B} := B^* \hat{A}^{-1} B$$

on V whose signature we denote by

$$\sigma(A, B) = \text{sign}(S_{A,B}). \quad (34)$$

We now consider maps $A: \mathbb{R} \rightarrow \mathcal{S}(W, H)$ and $B: \mathbb{R} \rightarrow \mathcal{L}(V, H)$ which are continuous with respect to the norm topology such that the limits

$$\lim_{s \rightarrow \pm\infty} A(s) = A^\pm, \quad \lim_{s \rightarrow \pm\infty} B(s) = B^\pm$$

exist and $A^\pm \in \mathcal{S}(W, H)$. Accordingly we define the map $A_B: \mathbb{R} \rightarrow \mathcal{S}(W \oplus V, H \oplus V)$ by

$$A_B(s) := A(s)_{B(s)}, \quad s \in \mathbb{R}.$$

Denote by

$$\mathcal{A} = \mathcal{A}(\mathbb{R}, W, H)$$

the space of maps $A: \mathbb{R} \rightarrow \mathcal{S}(W, H)$ as above, which in addition satisfy that A^\pm is bijective. Recall the following theorem of Robbin and Salamon about the existence of the spectral flow [40, Theorem 4.3].

Theorem B.4 *There exist unique maps $\mu: \mathcal{A}(\mathbb{R}, W, H) \rightarrow \mathbb{Z}$, one for every compact dense injection of Hilbert spaces $W \hookrightarrow H$, satisfying the following axioms.*

(homotopy) μ is constant on connected components of $\mathcal{A}(\mathbb{R}, W, H)$.

(constant) If A is constant, then $\mu(A) = 0$.

(direct sum) $\mu(A_1 \oplus A_2) = \mu(A_1) + \mu(A_2)$.

(normalization) For $W = H = \mathbb{R}$ and $A(t) = \arctan(t)$, we have $\mu(A) = 1$.

The number $\mu(A)$ is called the **spectral flow** of A .

These axioms easily imply the following generalization of the (normalization) axiom:

(crossing) For $W = H$ finite dimensional,

$$\mu(A) = \frac{1}{2} \left(\text{sign}(A^+) - \text{sign}(A^-) \right).$$

To define the spectral flow also for maps A whose limits A^\pm are not necessarily bijective we choose a smooth cutoff function $\beta \in C^\infty(\mathbb{R}, [-1, 1])$ such that $\beta(s) = 1$ for $s \geq 1$ and $\beta(s) = -1$ for $s \leq -1$ and define

$$A_\delta := A - \delta \beta \cdot \text{id}, \quad \mu(A) := \lim_{\delta \searrow 0} \mu(A_\delta). \quad (35)$$

Note that the limit in $\mu(A)$ stabilizes for sufficiently small $\delta > 0$.

Theorem B.5 *Let $A: \mathbb{R} \rightarrow \mathcal{S}(W, H)$ and $B: \mathbb{R} \rightarrow \mathcal{L}(V, H)$ be continuous maps whose limits $\lim_{s \rightarrow \pm\infty} A(s) = A^\pm$ and $\lim_{s \rightarrow \pm\infty} B(s) = B^\pm$ exist. Assume moreover that (A^\pm, B^\pm) are regular pairs. Then the spectral flows of A and A_B are related by*

$$\mu(A_B) = \mu(A) + \frac{1}{2} \left(\sigma(A^-, B^-) - \sigma(A^+, B^+) \right).$$

Proof: Choose two cutoff functions $\beta^\pm \in C^\infty(\mathbb{R}, [0, 1])$ with the property that $\beta^+(s) = 1$ for $s \geq 1$, $\beta^+(s) = 0$ for $s \leq 0$, $\beta^-(s) = 1$ for $s \leq -1$, and $\beta^-(s) = 0$ for $s \geq 0$. Define $S_{A,B} \in \mathcal{A}(\mathbb{R}, V, V)$ by

$$S_{A,B} = \beta^+ \cdot S_{A^+, B^+} + \beta^- \cdot S_{A^-, B^-}.$$

Abbreviate $P_V: H \oplus V \rightarrow V$ the canonical projection. We prove the Theorem in three steps.

Step 1: If $\delta \neq 0$ is small enough then $(A_B)_\delta$ is homotopic to $(A_\delta)_B - \delta P_V^* S_{A,B} P_V$.

To see this, note that $(A_B)_\delta = (A_\delta)_B - \delta P_V^* P_V$. So it suffices to show that for $\delta > 0$ sufficiently small and symmetric linear maps $S^\pm \in \mathcal{L}(V)$ whose norm is small enough the operators $(A_\delta^\pm)_{B^\pm} + P_V^* S^\pm P_V$ are bijective. By Theorem V.4.3 in [27] the operators $(A_\delta^\pm)_{B^\pm} + P_V^* S^\pm P_V$ are selfadjoint with dense domain $W \oplus V$, hence by Lemma B.1 their spectrum consists of eigenvalues. Thus it suffices to show injectivity. Let $(w, v) \in W \times V$ be in the kernel of $(A_\delta^\pm)_{B^\pm} + P_V^* S^\pm P_V$. Then (w, v) solves

$$\left. \begin{aligned} (A^\pm - \delta \cdot \text{id})w + B^\pm v &= 0 \\ (B^\pm)^* w + S^\pm v &= 0 \end{aligned} \right\} \quad (36)$$

which is equivalent to

$$\left. \begin{aligned} w &= -(A^\pm - \delta \cdot \text{id})^{-1} B^\pm v \\ -(B^\pm)^* (A^\pm - \delta \cdot \text{id})^{-1} B^\pm v + S^\pm v &= 0. \end{aligned} \right\} \quad (37)$$

But $(B^\pm)^* (A^\pm - \delta \cdot \text{id})^{-1} B^\pm$ converges to the nondegenerate linear map S_{A^\pm, B^\pm} as δ goes to zero, and hence the second equation in (37) has only the trivial solution $v = 0$ if the norm of S^\pm is small enough and hence $(w, v) = (0, 0)$. This shows injectivity and hence the assertion of Step 1 follows.

Step 2: For $\delta > 0$ small enough and $\epsilon \in [0, 1]$ the maps $(A_\delta)_{\epsilon B} - \delta P_V^* S_{A, B} P_V$ are in $\mathcal{A}(\mathbb{R}, W \oplus V, H \oplus V)$, i.e. their asymptotics are bijective.

Step 2 follows by a similar reasoning as Step 1. Assume that $(w, v) \in W \oplus V$ lies in the kernel of one of the asymptotic operators. Then (w, v) solves

$$\left. \begin{aligned} w &= -\epsilon (A^\pm - \delta \cdot \text{id})^{-1} B^\pm v \\ -\epsilon^2 (B^\pm)^* (A^\pm - \delta \cdot \text{id})^{-1} B^\pm v - \delta S_{A^\pm, B^\pm} v &= 0 \end{aligned} \right\}.$$

Since both terms in the second equation have the same sign and the first term converges to $-\epsilon^2 S_{A^\pm, B^\pm} v$ as δ goes to zero, these equations have only the trivial solution.

Step 3: We prove the theorem.

Using the properties of the spectral flow from Theorem B.4 we are now in position to compute

$$\begin{aligned} \mu((A_B)_\delta) &= \mu((A_\delta)_B - \delta P_V^* S_{A, B} P_V) \\ &= \mu((A_\delta)_0 - \delta P_V^* S_{A, B} P_V) \\ &= \mu(A_\delta \oplus -\delta S_{A, B}) \\ &= \mu(A_\delta) + \mu(-\delta S_{A, B}) \\ &= \mu(A_\delta) + \frac{1}{2} \left(\sigma(A^-, B^-) - \sigma(A^+, B^+) \right). \end{aligned}$$

Here we have used Step 1 for the first equality, Step 2 for the second one, and the (crossing) property of μ for the last one. Taking the limit $\delta \searrow 0$ the theorem

follows. □

There are scenarios where the signature $\sigma(A, B)$ can easily be computed. We formulate such an example for a finite dimensional Lagrange multiplier functional which can easily be generalized to infinite dimensional examples.

Lemma B.6 *Suppose that (M, g) is a Riemannian manifold, $f \in C^\infty(M)$, and $h \in C^\infty(M)$ such that 0 is a regular value of h . Let (x_0, v_0) be a critical point of the Lagrange multiplier functional $F \in C^\infty(M \times \mathbb{R})$ given by $F(x, v) = f(x) + v \cdot h(x)$. Assume that there exists $\epsilon > 0$ and a smooth curve $(x, v) \in C^\infty((-\epsilon, \epsilon), M \times \mathbb{R})$ satisfying $(x(0), v(0)) = (x_0, v_0)$ such that the following holds*

- (i) $\partial_\rho x(0) = \nabla h(x_0)$,
- (ii) $\partial_\rho v(0) \neq 0$,
- (iii) $(x(\rho), v(\rho))$ for $\rho \in (-\epsilon, \epsilon)$ is a critical point of the Lagrange multiplier functional $F^\rho \in C^\infty(M \times \mathbb{R})$ given by $F^\rho(x, v) := f(x) + v \cdot (h(x) - \rho)$.

Then the pair $(\text{Hess}_{F_{v_0}}(x_0), \nabla h(x_0))$ is regular in the sense of Definition B.3 and its signature is

$$\sigma(\text{Hess}_{F_{v_0}}(x_0), \nabla h(x_0)) = -\text{sign}(\partial_\rho v(0)).$$

Proof: The identity

$$dF^\rho(x(\rho), v(\rho)) = 0$$

for $\rho \in (-\epsilon, \epsilon)$ is equivalent to

$$\left. \begin{aligned} df(x(\rho)) + v(\rho) \cdot dh(x(\rho)) &= 0 \\ h(x(\rho)) &= \rho. \end{aligned} \right\} \quad (38)$$

The first equation in (38) can be written as

$$\nabla f(x(\rho)) + v(\rho) \cdot \nabla h(x(\rho)) = 0.$$

Differentiating this identity with respect to ρ and evaluating at $\rho = 0$ we compute using assumption (i)

$$\begin{aligned} 0 &= \text{Hess}_f(x_0) \partial_\rho x(0) + v_0 \text{Hess}_h(x_0) \partial_\rho x(0) + \partial_\rho v(0) \nabla h(x_0) \\ &= \text{Hess}_{F_{v_0}}(x_0) \nabla h(x_0) + \partial_\rho v(0) \nabla h(x_0). \end{aligned}$$

In particular, $\nabla h(x_0)$ is an eigenvector of $\text{Hess}_{F_{v_0}}$ to the nonzero eigenvalue $-\partial_\rho v(0)$.

It is now straightforward to check that the pair $(\text{Hess}_{F_{v_0}}(x_0), \nabla h(x_0))$ is regular. Condition (i) in Definition B.3 follows from the assumption that 0 is a regular

value of h and thus $\nabla h(x_0) \neq 0$. Since $\nabla h(x_0)$ is an eigenvector of the Hessian to a nonzero eigenvalue, condition (ii) is satisfied as well.

To compute the signature we calculate

$$\begin{aligned} \sigma(\text{Hess}_{F_{v_0}}(x_0), \nabla h(x_0)) &= \text{sign}(dh(x_0)\hat{\text{Hess}}_{F_{v_0}}(x_0)^{-1}\nabla h(x_0)) \\ &= \text{sign}\left(-\frac{\|\nabla h(x_0)\|^2}{\partial_\rho v(0)}\right) \\ &= -\text{sign}(\partial_\rho v(0)). \end{aligned}$$

This proves the lemma. \square

C Some topological obstructions

During the first author's talk at the Workshop on Symplectic Geometry, Contact Geometry and Interactions in Lille 2007, E. Giroux suggested that in the case $n = 2$, Corollary 1.7 results from the following topological fact.

Lemma C.1 *There exists no smooth embedding of $S^*S^2 \cong \mathbb{R}P^3$ into a subcritical Stein surface.*

Right after the talk, participants suggested the following three proofs of this fact. Note that every subcritical Stein surface is \mathbb{C}^2 or a boundary connected sum of copies of $S^1 \times \mathbb{R}^3$, which embeds smoothly into \mathbb{R}^4 ; thus for the lemma it suffices to prove that $\mathbb{R}P^3$ admits no embedding into \mathbb{R}^4 .

Proof 1 (V. Kharlamov): This proof is based on the following theorem of Whitney (see e.g. [29]): The Euler number $e(\Sigma) \in \mathbb{Z}$ of the normal bundle of a closed connected non-orientable surface Σ embedded in \mathbb{R}^4 satisfies $e(\Sigma) \equiv 2\chi(\Sigma) \pmod{4}$. Now suppose we have an embedding $\mathbb{R}P^3 \subset \mathbb{R}^4$. Then the normal Euler number of the linear subspace $\mathbb{R}P^2 \subset \mathbb{R}P^3 \subset \mathbb{R}^4$ satisfies $e(\mathbb{R}P^2) \equiv 2 \pmod{4}$. But a nonvanishing normal vector field to $\mathbb{R}P^3$ in \mathbb{R}^4 (which exists because $\mathbb{R}P^3$ is orientable) provides a nonvanishing section of the normal bundle of $\mathbb{R}P^2 \subset \mathbb{R}^4$, contradicting nontriviality of $e(\mathbb{R}P^2)$. \square

Proof 2 (T. Ekholm): This proof is based on the following theorem of Ekholm [11]: The Euler characteristic of the (resolved) self-intersection surface of a generic immersion of S^3 to \mathbb{R}^4 has the same parity as the number of quadruple points. Now suppose we have an embedding $\mathbb{R}P^3 \subset \mathbb{R}^4$. Composition with the covering $S^3 \rightarrow \mathbb{R}P^3$ yields an immersion of S^3 to \mathbb{R}^4 which can be perturbed (via a normal vector field to $\mathbb{R}P^3$ vanishing along $\mathbb{R}P^2$) to have self-intersection surface $\mathbb{R}P^2$ and no quadruple points, contradicting Ekholm's theorem. \square

The third proof, suggested by P. Lisca, yields in fact the following more general result:

Proposition C.2 *For $n \geq 2$ even there exists no smooth embedding of S^*S^n into a subcritical Stein $2n$ -manifold.*

Proof 3 (P. Lisca): Suppose we have an embedding $S^*S^n \cong \Sigma \subset V$ into a subcritical Stein $2n$ -manifold V for $n \geq 2$ even. From $H_{2n-1}(V; \mathbb{Z}) = 0$ it follows that Σ bounds a compact subset $B \subset V$. Denote by C the closure of $V \setminus B$, so $V = B \cup_{\Sigma} C$. Since n is even, the Gysin homology sequence of the sphere bundle $S^*S^n \rightarrow S^n$ shows $H_n(S^*S^n; \mathbb{Z}) = 0$ and $H_{n-1}(S^*S^n; \mathbb{Z}) = \mathbb{Z}_2$. Using this and $H_n(V; \mathbb{Z}) = 0$, the Mayer-Vietoris sequence for $V = B \cup_{\Sigma} C$ implies $H_n(B; \mathbb{Z}) = H_n(C; \mathbb{Z}) = 0$. Let $X := D^*S^n \cup_{\Sigma} B$ be the closed oriented $2n$ -manifold obtained by gluing the unit disk cotangent bundle D^*S^n (with orientation reversed) to B along Σ . Again by the Mayer-Vietoris sequence, we find that the free part $H_n(X; \mathbb{Z})/\text{torsion}$ is isomorphic to \mathbb{Z} and generated by the zero section $S^n \subset D^*S^n$. But for n even the zero section S^n has self-intersection number 2 in D^*S^n , hence -2 in X , contradicting unimodularity of the intersection form (which is an immediate consequence of Poincaré duality). \square

Remark. The fact that S^*S^2 has no *exact contact* embedding into a subcritical Stein surface V can also be proved symplectically as follows, using a deep result by Gromov about holomorphic fillings. Since every subcritical Stein surface admits an *exact symplectic* embedding into \mathbb{R}^4 , it suffices again to consider the case $V = \mathbb{R}^4$. Suppose there exists an exact contact embedding $\iota : S^*S^2 \hookrightarrow \mathbb{R}^4$. Removing the bounded component of $\mathbb{R}^4 \setminus \iota(S^*S^2)$ and gluing in the unit ball bundle D^*S^2 yields an exact convex symplectic manifold W which contains an embedded Lagrangian 2-sphere (the zero section in D^*S^2). On the other hand, W is symplectomorphic to \mathbb{R}^4 outside compact set. So a result of Gromov [24] implies that W is in fact symplectomorphic to \mathbb{R}^4 . But this is a contradiction because \mathbb{R}^4 does not admit any embedded Lagrangian 2-spheres.

Remark. We have not investigated obstructions to smooth embeddings of S^*S^n into subcritical Stein manifolds for n odd. As pointed out in the introduction, at least for $n = 3$ and $n = 7$ there are no obstructions and S^*S^n embeds smoothly into \mathbb{C}^n .

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